Operators on Differential Forms

In the following

- $\{U, \zeta\}$ is a coordinate patch for a Riemann surface M,
- $f(x,y),\ g(x,y), u(x,u),\ v(x,y)$ are functions on $\zeta(U) \subset \mathbb{R}^2$
- F is a 0-form on M,
- ω is a 1-form on M with

$$\begin{aligned} \omega\big|_{\{U,\zeta\}} &= f(x,y)\,dx + g(x,y)\,dy \\ &= u(x,y)\,dz + v(x,y)\,d\bar{z} \end{aligned}$$

- Ω is a 2-form on M

We shall also use the complex notation

$$dz = dx + idy \qquad\qquad d\bar{z} = dx - idy$$

$$u_z(x,y) = \tfrac{1}{2}\big(\tfrac{\partial}{\partial x} - i\tfrac{\partial}{\partial y}\big)u(x,y) \quad u_{\bar{z}}(x,y) = \tfrac{1}{2}\big(\tfrac{\partial}{\partial x} + i\tfrac{\partial}{\partial y}\big)u(x,y)$$

The Differential Operator d

For C^1 forms

$$\begin{aligned} dF\big|_{\{U,\zeta\}} &= \tfrac{\partial}{\partial x}\big(F\circ\zeta^{-1}\big)(x,y)\,dx + \tfrac{\partial}{\partial y}\big(F\circ\zeta^{-1}\big)(x,y)\,dy \\ &= \big(F\circ\zeta^{-1}\big)_z(x,y)\,dz + \big(F\circ\zeta^{-1}\big)_{\bar{z}}(x,y)\,d\bar{z} \\ d\omega\big|_{\{U,\zeta\}} &= \big[\tfrac{\partial g}{\partial x}(x,y) - \tfrac{\partial f}{\partial y}(x,y)\big]\,dx\wedge dy \\ &= \big[v_z(x,y) - u_{\bar{z}}(x,y)\big]\,dz\wedge d\bar{z} \\ d\Omega &= 0 \end{aligned}$$

The Differential Operators $\partial, \bar{\partial}$

For C^1 forms

$$\begin{aligned} \partial F\big|_{\{U,\zeta\}} &= \tfrac{1}{2}\big(\tfrac{\partial}{\partial x} - i\tfrac{\partial}{\partial y}\big)\big(F\circ\zeta^{-1}\big)(x,y)\,dz = \big(F\circ\zeta^{-1}\big)_z\,dz \\ \bar{\partial} F\big|_{\{U,\zeta\}} &= \tfrac{1}{2}\big(\tfrac{\partial}{\partial x} + i\tfrac{\partial}{\partial y}\big)\big(F\circ\zeta^{-1}\big)(x,y)\,d\bar{z} = \big(F\circ\zeta^{-1}\big)_{\bar{z}}\,d\bar{z} \\ \partial\omega\big|_{\{U,\zeta\}} &= v_z(x,y)\,dz\wedge d\bar{z} \\ \bar{\partial}\omega\big|_{\{U,\zeta\}} &= -u_{\bar{z}}(x,y)\,dz\wedge d\bar{z} \\ \partial\Omega &= 0 \\ \bar{\partial}\Omega &= 0 \end{aligned}$$

The Conjugation Operator $*$

$$
\begin{aligned}
*\omega\Big|_{\{U,\zeta\}} &= -g(x,y)\,dx + f(x,y)\,dy \\
&= -iu(x,y)\,dz + iv(x,y)d\bar{z}
\end{aligned}
$$

Complex Conjugation

$$
\begin{aligned}
\bar{\omega}\Big|_{\{U,\zeta\}} &= \overline{f(x,y)}\,dx + \overline{g(x,y)}\,dy \\
&= \overline{v(x,y)}\,dz + \overline{u(x,y)}\,d\bar{z}
\end{aligned}
$$

Various Identities

(1) $$dx = \tfrac{1}{2}(dz + d\bar{z})$$

(2) $$dy = \tfrac{1}{2i}(dz - d\bar{z})$$

(3) $$dz \wedge d\bar{z} = -2idx \wedge dy$$

(4) $$dz' = \tfrac{dz'}{dz}\,dz \quad d\bar{z}' = \overline{\tfrac{dz'}{dz}}\,d\bar{z}$$

(5) $$dz' \wedge d\bar{z}' = \left|\tfrac{dz'}{dz}\right|^2 dz \wedge d\bar{z}$$

(6) $$\bar{u}_z(x,y) = \overline{u_{\bar{z}}(x,y)} \quad \bar{u}_{\bar{z}}(x,y) = \overline{u_z(x,y)}$$

(7) If $u(x,y) = w(x+iy)$ with w analytic, $u_z(x,y) = w'(x+iy) \quad u_{\bar{z}}(x,y) = 0$

(8) If $u(x,y) = w(x-iy)$ with w analytic, $u_{\bar{z}}(x,y) = w'(x-iy) \quad u_z(x,y) = 0$

(9) $$d^2 = 0$$

(10) $$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

(11) $$d = \partial + \bar{\partial}$$

(12) $$** = -1$$

(13) $$\overline{dF} = d\bar{F}$$

(14) $$\overline{*\omega} = *\bar{\omega}$$

(15) $$\partial\omega = (d + d*)\omega \quad \bar{\partial}\omega = (d - d*)\omega$$

(16) $$\Delta F = d * dF = -2i\bar{\partial}\partial F$$