Elliptic Regularity

Let Ω be an open subset of $\mathbb{R}^d$. A measurable, locally square integrable function φ is said to be a weak solution of Laplace's equation in Ω if

$$\iint_\Omega \varphi(\vec{\mathbf{r}})\, \Delta\eta(\vec{\mathbf{r}})\ d^d\vec{\mathbf{r}} = 0$$

for all C_0^∞ functions η that are supported in Ω. The theorem that any weak solution of an elliptic partial differential equation in Ω is C^∞ (technically, equal almost everywhere in Ω to a C^∞ function) is called elliptic regularity. In this course, we are interested in harmonic functions in $d = 2$, so we now prove elliptic regularity for Laplace's equation in $d = 2$.

Theorem. *Let Ω be an open subset of $\mathbb{R}^2$. Let φ be a measurable, locally square integrable function that is a weak solution of Laplace's equation in Ω. Then φ is equal almost everywhere in Ω to a C^∞ function.*

Motivation for proof: By way of motivation for the strategy that we'll use to prove this Theorem, I'll first outline a simple proof that any C^2 function φ that obeys $\Delta\varphi = 0$ is in fact C^∞. Recall that, by the Cauchy integral formula, any analytic function, $f(z)$, obeys

$$f(z') = \tfrac{1}{2\pi i} \int_{|z-z'|=r} \tfrac{f(z)}{z-z'} dz$$

Parametrizing the circle $|z - z'| = r$ by $z = z' + re^{i\theta}$,

$$f(z') = \tfrac{1}{2\pi i} \int_0^{2\pi} \tfrac{f(z'+re^{i\theta})}{re^{i\theta}}\ ire^{i\theta} d\theta = \tfrac{1}{2\pi} \int_0^{2\pi} f(z' + re^{i\theta})\ d\theta$$

This is called the "Mean–value Property". It also holds for harmonic functions. That is, if $\Delta\varphi = 0$, then

$$\varphi(x', y') = \tfrac{1}{2\pi} \int_0^{2\pi} \varphi\big((x', y') + r(\cos\theta, \sin\theta)\big)\ d\theta$$

This is proven using Green's Theorem, which is the same way that the Cauchy Integral Theorem is proven. Now let $g \in C_0^\infty\big([0,\infty)\big)$ obey $\int_0^\infty g(r) r dr = \frac{1}{2\pi}$. Then

$$\begin{aligned}
\varphi(x', y') &= \int_0^\infty dr\ r\ g(r)\ 2\pi\varphi(x', y') \\
&= \int_0^\infty dr\ r \int_0^{2\pi} d\theta\ g(r)\varphi\big((x', y') + r(\cos\theta, \sin\theta)\big) \\
&= \iint dxdy\ g(\|(x, y)\|)\varphi\big((x', y') + (x, y)\big) \\
&= \iint dxdy\ g(\|(x' - x, y' - y)\|)\varphi\big((x, y)\big)
\end{aligned}$$

The right hand side is trivially C^∞ because all derivatives with respect to x' or y' act on $g(\|(x'-x, y'-y)\|)$, which is C^∞ because the length $\|(x'-x, y'-y)\|$ is C^∞ in (x', y') except at $x'-x = y'-y = 0$ and $g(r)$ is C^∞ and vanishes for r in a neighbourhood of 0.

Proof: Every open set is a union of open disks. That φ is locally square integrable in Ω means that φ is square integrable on some neighbourhood of each point of Ω. So we may choose the disks so that φ is L^2 on each disk. Thus it suffices to consider Ω's that are open disks. By translating and scaling, it suffices to consider the unit disk centred on the origin, which we denote D, and we may assume that φ is L^2 on D.

We first construct the function that is going to play the role of g in the motivation above. Let $\vec{\mathbf{r}} = (x, y)$. We shall exploit two properties of the function $\ln\|\vec{\mathbf{r}}\|$. The first is that $\ln\|\vec{\mathbf{r}}\|$ is defined and harmonic for all $\vec{\mathbf{r}} \neq \mathbf{0}$. This is shown by the computation

$$d\ln\|\vec{\mathbf{r}}\| = \tfrac{1}{2}d\ln(x^2+y^2) = \tfrac{xdx+ydy}{x^2+y^2}$$

$$\Delta\ln\|\vec{\mathbf{r}}\| = d * d\ln\|\vec{\mathbf{r}}\| = d\tfrac{-ydx+xdy}{x^2+y^2} = \tfrac{2(x^2+y^2)dx\wedge dy-(2xdx+2ydy)\wedge(-ydx+xdy)}{(x^2+y^2)^2} = 0 \quad \text{(P1)}$$

The second property of $\ln\|\vec{\mathbf{r}}\|$ that we shall use is the following. Let C_δ be the circle of radius δ centered on $\mathbf{0}$, oriented, as usual, in the counterclockwise direction. Then, for any continuous function $\psi(\vec{\mathbf{r}})$,

$$\lim_{\delta\to 0+}\oint_{C_\delta}\psi(\vec{\mathbf{r}})\ *d\ln\|\vec{\mathbf{r}}\| = 2\pi\psi(\mathbf{0}) \quad \text{(P2)}$$

To see this, parametrize C_δ by $\vec{\mathbf{r}}(t) = \big(x(t), y(t)\big) = \delta\big(\cos t, \sin t\big)$ with $0 \le t \le 2\pi$. When we evaluate the integral $\oint_{C_\delta}\psi(\vec{\mathbf{r}})\ *d\ln\|\vec{\mathbf{r}}\|$ using this parametrization, $*d\ln\|\vec{\mathbf{r}}\| = \frac{-ydx+xdy}{x^2+y^2}$ is replaced by

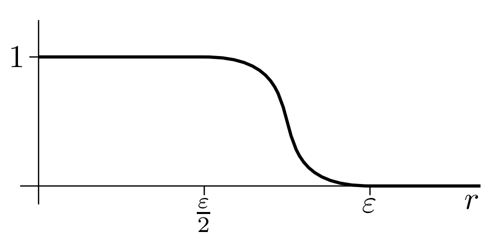
$$\tfrac{-y(t)x'(t)dt+x(t)y'(t)dt}{x(t)^2+y(t)^2} = dt$$

so that, using the continuity of ψ,

$$\begin{aligned}\lim_{\delta\to 0+}\oint_{C_\delta}\psi(\vec{\mathbf{r}})\ *d\ln\|\vec{\mathbf{r}}\| &= \lim_{\delta\to 0+}\int_0^{2\pi}\psi(\delta\cos t, \delta\sin t)\,dt \\ &= \int_0^{2\pi}\lim_{\delta\to 0+}\psi(\delta\cos t, \delta\sin t)\,dt = 2\pi\psi(\mathbf{0})\end{aligned}$$

Now we use $\frac{1}{2\pi}\ln\|\vec{\mathbf{r}}\|$ to build the function that plays the role of g. Let $0 < \varepsilon \ll 1$ and let ρ be a C^∞ function on $[0, \infty)$ that obeys

$$\begin{aligned} &\rho(r) = 1 && \text{for } 0 \le r \le \tfrac{\varepsilon}{2} \\ &0 \le \rho(r) \le 1 && \text{for } \tfrac{\varepsilon}{2} \le r \le \varepsilon \\ &\rho(r) = 0 && \text{for } r \ge \varepsilon \end{aligned}$$



Define

$$\omega(\vec{\mathbf{r}}) = \tfrac{1}{2\pi}\rho\big(\|\vec{\mathbf{r}}\|\big)\ln\|\vec{\mathbf{r}}\|$$

$$\gamma(\vec{\mathbf{r}}) = \begin{cases} -\big(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\big)\omega(\vec{\mathbf{r}}) & \text{if } \vec{\mathbf{r}} \ne \mathbf{0} \\ 0 & \text{if } \vec{\mathbf{r}} = \mathbf{0} \end{cases}$$

$$\Phi(\vec{\mathbf{r}}') = \iint_D \gamma(\vec{\mathbf{r}}' - \vec{\mathbf{r}})\,\varphi(\vec{\mathbf{r}})\; dx \wedge dy$$

Note that

- ∘ $\omega(\vec{\mathbf{r}})$ is defined and C^∞ for all $\vec{\mathbf{r}} \ne 0$.
- ∘ $\omega(\vec{\mathbf{r}})$ is supported on $\|\vec{\mathbf{r}}\| \le \varepsilon$.
- ∘ $\omega(\vec{\mathbf{r}}) = \frac{1}{2\pi}\ln\|\vec{\mathbf{r}}\|$ for $0 < \|\vec{\mathbf{r}}\| < \frac{\varepsilon}{2}$ so that $\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)\omega(\vec{\mathbf{r}})$ vanishes for $0 < \|\vec{\mathbf{r}}\| < \frac{\varepsilon}{2}$
- ∘ $\gamma(\vec{\mathbf{r}})$ is defined and C^∞ on all of $\mathbb{R}^2$.
- ∘ $\gamma(\vec{\mathbf{r}})$ is supported on $\|\vec{\mathbf{r}}\| \le \varepsilon$.
- ∘ $\Phi(\vec{\mathbf{r}}')$ is defined and C^∞ on all of $\mathbb{R}^2$ since γ is C^∞ and φ is L^1 on D.

The Theorem now follows from part b of the Lemma below, which implies that $\varphi(\vec{\mathbf{r}}') = \Phi(\vec{\mathbf{r}}')$ for almost all $\vec{\mathbf{r}}$ with $\|\vec{\mathbf{r}}\| \le 1 - 2\varepsilon$. ■

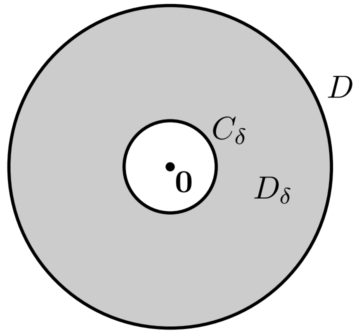
More motivation: To motivate the choice of γ above, I'll now show that if φ is harmonic, that is, if φ is C^2 and obeys $\Delta\varphi = 0$, then $\Phi(\vec{\mathbf{r}}') = \varphi(\vec{\mathbf{r}}')$ for all $|\vec{\mathbf{r}}'| < 1 - \varepsilon$. First observe that, since $|\vec{\mathbf{r}}'| < 1-\varepsilon$ and $\gamma(\vec{\mathbf{r}}' - \vec{\mathbf{r}})$ vanishes for $\|\vec{\mathbf{r}}' - \vec{\mathbf{r}}\| \ge \varepsilon$, $\gamma(\vec{\mathbf{r}}' - \vec{\mathbf{r}})$ vanishes unless $\vec{\mathbf{r}} \in D$. Thus

$$\begin{aligned}\Phi(\vec{\mathbf{r}}') &= \iint_D \gamma(\vec{\mathbf{r}}' - \vec{\mathbf{r}})\,\varphi(\vec{\mathbf{r}})\; dx \wedge dy \quad = \iint_{\mathbb{R}^2} \gamma(\vec{\mathbf{r}}' - \vec{\mathbf{r}})\,\varphi(\vec{\mathbf{r}})\; dx \wedge dy \\ &= \iint_{\mathbb{R}^2} \gamma(-\vec{\mathbf{r}})\,\varphi(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\; dx \wedge dy = \iint_{\mathbb{R}^2} \gamma(\vec{\mathbf{r}})\,\varphi(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\; dx \wedge dy\end{aligned}$$

since γ is even. We are now going to substitute in (for $\vec{\mathbf{r}} \ne 0$) $\gamma(\vec{\mathbf{r}}) = -\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)\omega(\vec{\mathbf{r}})$ and integrate by parts a couple of times. To treat the singularity in ω at $\vec{\mathbf{r}} = 0$ carefully, we eliminate 0 from the domain of integration. Since γ and φ are both continuous at $\vec{\mathbf{r}} = 0$ and since $\gamma(\vec{\mathbf{r}})$ vanishes unless $\|\vec{\mathbf{r}}\| \le \varepsilon < 1$,

$$\begin{aligned}\Phi(\vec{\mathbf{r}}') &= \lim_{\delta\to 0}\iint_{\|\vec{\mathbf{r}}\|\ge\delta} \gamma(\vec{\mathbf{r}})\,\varphi(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\; dx \wedge dy = \lim_{\delta\to 0}\iint_{D_\delta} \gamma(\vec{\mathbf{r}})\,\varphi(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\; dx \wedge dy \\ &= \lim_{\delta\to 0} -\iint_{D_\delta} \varphi(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\Delta\omega(\vec{\mathbf{r}})\end{aligned}$$

where $D_\delta = \big\{\, \vec{\mathbf{r}} \in \mathbb{R}^2 \;\big|\; \delta \le \|\vec{\mathbf{r}}\| \le 1 \,\big\}$ is the unit disk with the disk of radius δ removed. By



Green's formula (number 6 on our list of integration formulae)

$$\iint_{D_\delta} \omega(\vec{\mathbf{r}})\Delta\varphi(\vec{\mathbf{r}}+\vec{\mathbf{r}}') - \iint_{D_\delta} \varphi(\vec{\mathbf{r}}+\vec{\mathbf{r}}')\Delta\omega(\vec{\mathbf{r}}) = \int_{\delta D_\delta} \omega(\vec{\mathbf{r}}) * d\varphi(\vec{\mathbf{r}}+\vec{\mathbf{r}}') - \int_{\delta D_\delta} \varphi(\vec{\mathbf{r}}+\vec{\mathbf{r}}') * d\omega(\vec{\mathbf{r}})$$

The first term on the left hand side vanishes because φ is harmonic. The boundary $\delta D_\delta = C_1 - C_\delta$. The minus sign is there because the inside part of the boundary of δD is oriented in the opposite direction to C_δ. The outer, C_1, part of the boundary integrals are zero because $\omega(\vec{\mathbf{r}})$ vanishes for all $\|\vec{\mathbf{r}}\| > \varepsilon$. Furthermore, if $\delta < \frac{\varepsilon}{2}$, $\omega(\vec{\mathbf{r}}) = \frac{1}{2\pi}\log\|\vec{\mathbf{r}}\|$ on the inner part, C_δ, of the boundary. So

$$\Phi(\vec{\mathbf{r}}') = \lim_{\delta\to 0} \tfrac{1}{2\pi} \oint_{C_\delta} \varphi(\vec{\mathbf{r}}+\vec{\mathbf{r}}') * d\log\|\vec{\mathbf{r}}\| - \lim_{\delta\to 0} \tfrac{1}{2\pi} \oint_{C_\delta} \log\|\vec{\mathbf{r}}\| * d\varphi(\vec{\mathbf{r}}+\vec{\mathbf{r}}')$$

The first term on the right hand side is exactly $\varphi(\vec{\mathbf{r}}')$ by the delta function like property (P2). The second term on the right hand side vanishes. To see this, parametrize C_δ by $\vec{\mathbf{r}}(\theta) = \big(x(\theta), y(\theta)\big) = \delta(\cos\theta, \sin\theta)$ and observe that, because φ is C^2, $*d\varphi(\vec{\mathbf{r}}+\vec{\mathbf{r}}') = -\varphi_y \frac{dx}{d\theta}d\theta + \varphi_x \frac{dy}{d\theta}d\theta = \varphi_y\,\delta\sin\theta\,d\theta + \varphi_x\,\delta\cos\theta\,d\theta$ is some continous, and hence bounded function, times $\delta\,d\theta$. Consequently, the second term on the right hand side is bounded in magnitude by a constant times

$$\lim_{\delta\to 0} \tfrac{1}{2\pi} \int_0^{2\pi} \log\delta\ \delta\,d\theta = \lim_{\delta\to 0} \tfrac{1}{2\pi}(\log\delta)(2\pi\delta) = 0$$

Hence $\Phi(\vec{\mathbf{r}}') = \varphi(\vec{\mathbf{r}}')$ for all $\|\vec{\mathbf{r}}'\| < 1-\varepsilon$. In particular $\varphi(\vec{\mathbf{r}}')$ is C^∞ for all $\|\vec{\mathbf{r}}'\| < 1-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\varphi(\vec{\mathbf{r}}')$ is C^∞ for all $\|\vec{\mathbf{r}}'\| < 1$. This ends "More motivation".

We now need to consider functions of both $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}}'$. We use d' and Δ' to denote the operators d and Δ acting on functions of $\vec{\mathbf{r}}'$. For example

$$\begin{aligned} d\,f(x,y,x',y') &= \tfrac{\partial f}{\partial x}(x,y,x',y')\,dx + \tfrac{\partial f}{\partial y}(x,y,x',y')\,dy \\ d'\,f(x,y,x',y') &= \tfrac{\partial f}{\partial x'}(x,y,x',y')\,dx' + \tfrac{\partial f}{\partial y'}(x,y,x',y')\,dy' \end{aligned}$$

Lemma. *Let* $\mu(\vec{\mathbf{r}})$ *be* C^∞ *and supported in* $\|\vec{\mathbf{r}}\| \le 1-2\varepsilon$. *Define*

$$\eta(\vec{\mathbf{r}}') = \iint_D \omega(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\,\mu(\vec{\mathbf{r}})\ dx \wedge dy$$

then

a) $\Delta'\eta(\vec{\mathbf{r}}') = \big\{\mu(\vec{\mathbf{r}}') - \iint_D \gamma(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\,\mu(\vec{\mathbf{r}})\ dx \wedge dy\big\}\, dx' \wedge dy'$

b) $\iint_D \mu(\vec{\mathbf{r}})\big[\varphi(\vec{\mathbf{r}}) - \Phi(\vec{\mathbf{r}})\big]\ dx \wedge dy = 0$

Remark. Let $B_{1-2\varepsilon} = \big\{\ \vec{\mathbf{r}} \in \mathbb{R}^2 \bigm| \|\vec{\mathbf{r}}\| \le 1-2\varepsilon\ \big\}$. Since $C_0^\infty(B_{1-2\varepsilon})$ is dense in $L^2(B_{1-2\varepsilon})$ and $\overline{\varphi(\vec{\mathbf{r}}) - \Phi(\vec{\mathbf{r}})}$ is in $L^2(B_{1-2\varepsilon})$, part (b) has the consequence that $\iint_{B_{1-2\varepsilon}} \big|\varphi(\vec{\mathbf{r}}) - \Phi(\vec{\mathbf{r}})\big|^2\, dxdy = 0$ and hence that $\varphi(\vec{\mathbf{r}}) - \Phi(\vec{\mathbf{r}}) = 0$ almost everywhere on $B_{1-2\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of the Theorem.

Proof: b) We first prove part (b) assuming part (a). Since $\mu(\vec{\mathbf{r}})$ vanishes unless $|\vec{\mathbf{r}}| \le 1-2\varepsilon$ and $\omega(\vec{\mathbf{r}}' - \vec{\mathbf{r}})$ vanishes unless $\|\vec{\mathbf{r}}' - \vec{\mathbf{r}}\| \le \varepsilon$, $\eta(\vec{\mathbf{r}}')$ vanishes unless $\|\vec{\mathbf{r}}'\| \le 1-\varepsilon$. Furthermore, as ω is L^1 and μ is C^∞ and supported in D,

$$\eta(\vec{\mathbf{r}}') = \iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\,\mu(\vec{\mathbf{r}})\ dx \wedge dy = \iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}})\,\mu(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\ dx \wedge dy$$

is also C^∞ and hence is in $C_0^\infty(D)$. Our main assumpion is that φ is a weak solution of Laplace's equation in D. Hence

$$\begin{aligned}
0 &= \iint_D \varphi(\vec{\mathbf{r}}')\,\Delta\eta(\vec{\mathbf{r}}') \\
&= \iint_D \varphi(\vec{\mathbf{r}}')\Big\{\mu(\vec{\mathbf{r}}') - \iint_D \gamma(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\mu(\vec{\mathbf{r}})\ dx \wedge dy\Big\} dx' \wedge dy' \\
&= \iint_D \mu(\vec{\mathbf{r}}')\Big\{\varphi(\vec{\mathbf{r}}') - \iint_D \gamma(\vec{\mathbf{r}}' - \vec{\mathbf{r}})\varphi(\vec{\mathbf{r}})\ dx \wedge dy\Big\} dx' \wedge dy'
\end{aligned}$$

We made the change of variables $\vec{\mathbf{r}} \leftrightarrow \vec{\mathbf{r}}'$ in the second term. The right hand side is exactly $\iint_D \mu(\vec{\mathbf{r}})\big[\varphi(\vec{\mathbf{r}}) - \Phi(\vec{\mathbf{r}})\big]\ dx \wedge dy$.

a) Since μ is supported in D,

$$\eta(\vec{\mathbf{r}}') = \iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}} - \vec{\mathbf{r}}')\,\mu(\vec{\mathbf{r}})\ dx \wedge dy = \iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}})\,\mu(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\ dx \wedge dy$$

In the integrand, μ is a function of $\vec{\mathbf{r}} + \vec{\mathbf{r}}'$ only, so

$$\begin{aligned}
\Delta'\eta(\vec{\mathbf{r}}') &= dx' \wedge dy' \iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}})\,\big(\tfrac{\partial^2}{\partial x'^2} + \tfrac{\partial^2}{\partial y'^2}\big)\mu(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\ dx \wedge dy \\
&= dx' \wedge dy' \iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}})\,\big(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\big)\mu(\vec{\mathbf{r}} + \vec{\mathbf{r}}')\ dx \wedge dy \\
&= dx' \wedge dy' \iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}})\,\Delta\mu(\vec{\mathbf{r}} + \vec{\mathbf{r}}')
\end{aligned}$$

We now integrate by parts twice (apply Green's formula) twice, being careful about the singularity of ω at the origin. Since ω is supported in D and is in $L^1(D)$ and μ is C^∞,

$$\iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}})\,\Delta\mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}') = \lim_{\delta\to 0+} \iint_{D_\delta} \omega(\vec{\mathbf{r}})\,\Delta\mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}')$$

By Green's formula,

$$\iint_{D_\delta} \omega(\vec{\mathbf{r}})\Delta\mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}') - \iint_{D_\delta} \mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}')\Delta\omega(\vec{\mathbf{r}}) = \int_{\delta D_\delta} \omega(\vec{\mathbf{r}}) * d\mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}') - \int_{\delta D_\delta} \mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}') * d\omega(\vec{\mathbf{r}})$$

Again, the boundary $\delta D_\delta = C_1 - C_\delta$ and the outer, C_1, part of the boundary integrals are zero because $\omega(\vec{\mathbf{r}})$ vanishes for all $\|\vec{\mathbf{r}}\| > \varepsilon$. And the C_δ part of the first boundary integral again tends to zero with δ because, if $\delta < \frac{\varepsilon}{2}$, $\omega(\vec{\mathbf{r}}) = \frac{1}{2\pi}\log\|\vec{\mathbf{r}}\|$ on C_δ, both first derivatives of μ are bounded, say by K, and the circumference of C_δ is $2\pi\delta$ so that

$$\left| \oint_{C_\delta} \omega(\vec{\mathbf{r}}) \, * d\mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}') \right| \le \left(\tfrac{1}{2\pi}\ln\delta\right)(2K)(2\pi\delta)$$

On D_δ, $\Delta\omega(\vec{\mathbf{r}}) = -\gamma(\vec{\mathbf{r}})dx\wedge dy$, so that

$$\begin{aligned}
\iint_{\mathbb{R}^2} \omega(\vec{\mathbf{r}})\,\Delta\mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}') &= \lim_{\delta\to 0+} \oint_{C_\delta} \mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}') * d\omega(\vec{\mathbf{r}}) - \lim_{\delta\to 0+} \iint_{D_\delta} \mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}')\gamma(\vec{\mathbf{r}})\ dx\wedge dy \\
&= \mu(\vec{\mathbf{r}}') - \lim_{\delta\to 0+} \iint_{D_\delta} \mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}')\gamma(\vec{\mathbf{r}})\ dx\wedge dy \qquad \text{by (P2)} \\
&= \mu(\vec{\mathbf{r}}') - \iint_{D} \mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}')\gamma(\vec{\mathbf{r}})\ dx\wedge dy \\
&= \mu(\vec{\mathbf{r}}') - \iint_{\mathbb{R}^2} \mu(\vec{\mathbf{r}}+\vec{\mathbf{r}}')\gamma(\vec{\mathbf{r}})\ dx\wedge dy \\
&= \mu(\vec{\mathbf{r}}') - \iint_{\mathbb{R}^2} \mu(\vec{\mathbf{r}})\gamma(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\ dx\wedge dy \\
&= \mu(\vec{\mathbf{r}}') - \iint_{D} \gamma(\vec{\mathbf{r}}-\vec{\mathbf{r}}')\mu(\vec{\mathbf{r}})\ dx\wedge dy
\end{aligned}$$

since γ and μ are supported in D. ■