

Lattices and Periodic Functions

Definition L.1 Let $f(\mathbf{x})$ be a function on \mathbb{R}^d .

a) The vector $\boldsymbol{\gamma} \in \mathbb{R}^d$ is said to be a period for f if

$$f(\mathbf{x} + \boldsymbol{\gamma}) = f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^d$$

b) Set

$$\mathcal{P}_f = \{ \boldsymbol{\gamma} \in \mathbb{R}^d \mid \boldsymbol{\gamma} \text{ is a period for } f \}$$

If $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathcal{P}_f$ then $\boldsymbol{\gamma} + \boldsymbol{\gamma}' \in \mathcal{P}_f$ and if $\boldsymbol{\gamma} \in \mathcal{P}_f$ then $-\boldsymbol{\gamma} \in \mathcal{P}_f$ (sub $\mathbf{x} = \mathbf{z} - \boldsymbol{\gamma}$ into $f(\mathbf{x} + \boldsymbol{\gamma}) = f(\mathbf{x})$). Furthermore, the zero vector $\mathbf{0} \in \mathbb{R}^d$ is always in \mathcal{P}_f . Thus \mathcal{P}_f is a (commutative) group under addition and

$$\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_p \in \mathcal{P}_f \Rightarrow n_1\boldsymbol{\gamma}_1 + \dots + n_p\boldsymbol{\gamma}_p \in \mathcal{P}_f \quad \text{for all } p \in \mathbb{N} \text{ and } n_1, \dots, n_p \in \mathbb{Z}$$

Example L.2

- a) If $f(x, y) = \sin\left(\frac{2\pi x}{\ell_1}\right) \cos\left(\frac{2\pi y}{\ell_2}\right)$, then $\mathcal{P}_f = \{ (m\ell_1, n\ell_2) \mid m, n \in \mathbb{Z} \}$.
- b) If $f(x, y) = \sin\left(\frac{2\pi x}{\ell_1}\right)$, then $\mathcal{P}_f = \{ (m\ell_1, y) \mid m \in \mathbb{Z}, y \in \mathbb{R} \}$.
- c) If $f(x, y) = \sin\left(\frac{2\pi x}{\ell_1}\right) \sinh y$, then $\mathcal{P}_f = \{ (m\ell_1, 0) \mid m \in \mathbb{Z} \}$.

To exclude functions, as in Example L.2.b, that are constant in some direction, it suffices to require that $\mathbf{0}$ be an isolated point of \mathcal{P}_f . That is, to require that there be a number $r > 0$ such that every nonzero $\boldsymbol{\gamma} \in \mathcal{P}_f$ obeys $|\boldsymbol{\gamma}| \geq r$.

Proposition L.3 *If \mathcal{P} is an additive subgroup of \mathbb{R}^d and $\mathbf{0}$ is an isolated point of \mathcal{P} , then there are $d' \leq d$ and independent vectors $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_{d'} \in \mathbb{R}^d$ such that*

$$\mathcal{P} = \{ n_1\boldsymbol{\gamma}_1 + \dots + n_{d'}\boldsymbol{\gamma}_{d'} \mid n_1, \dots, n_{d'} \in \mathbb{Z} \}$$

Proof:

Claim 1. \mathcal{P} has a shortest nonzero element.

Proof of Claim 1: Define $r = \inf \{ |\gamma| \mid \gamma \in \mathcal{P}, \gamma \neq \mathbf{0} \}$. If there were no shortest element, there would be a sequence of vectors β_1, β_2, \dots in \mathcal{P} with $\lim_{i \rightarrow \infty} |\beta_i| = r$ and $r < |\beta_i| \leq 2r$ for every $i = 1, 2, \dots$. Because the closed ball of radius $2r$ is compact, the sequence has a limit point and hence has a Cauchy subsequence. In particular, there are β_i and β_j in the sequence, with $\beta_i \neq \beta_j$ with $|\beta_i - \beta_j| < \frac{r}{2}$. But this is impossible, because $\beta_i - \beta_j$ would be a nonzero element of \mathcal{P} with length smaller than r .

Claim 2. Let γ_1 be a shortest nonzero element of \mathcal{P} and set $\mathcal{P}_1 = \{ \gamma \in \mathcal{P} \mid \gamma \parallel \gamma_1 \}$. Then $\mathcal{P}_1 = \{ n\gamma_1 \mid n \in \mathbb{Z} \}$.

Proof of Claim 2: If $x\gamma_1 \in \mathcal{P}$ with x not an integer, then $(x - [x])\gamma_1$ (where $[\cdot]$ denotes integer part) is a nonzero element of \mathcal{P} with length strictly smaller than the length of γ_1 .

If $\mathcal{P} = \mathcal{P}_1$, we have finished. Otherwise continue with

Claim 3. Denote by \mathbb{P}_1 orthogonal projection in \mathbb{R}^d onto the line $\{ x\gamma_1 \mid x \in \mathbb{R} \}$ and by $\mathbb{P}_1^\perp = \mathbb{1} - \mathbb{P}_1$ orthogonal projection perpendicular to the line $\{ x\gamma_1 \mid x \in \mathbb{R} \}$. Then $\mathcal{P} \setminus \mathcal{P}_1$ has an element whose distance from the line $\{ x\gamma_1 \mid x \in \mathbb{R} \}$ is a minimum, i.e. that minimizes $|\mathbb{P}_1^\perp \gamma|$.

Proof of Claim 3: Define $r_1 = \inf \{ |\mathbb{P}_1^\perp \gamma| \mid \gamma \in \mathcal{P} \setminus \mathcal{P}_1 \}$. If there were no minimizing element, there would be a sequence of vectors β_1, β_2, \dots in \mathcal{P} with

$$2r_1 \geq |\mathbb{P}_1^\perp \beta_1| > |\mathbb{P}_1^\perp \beta_2| > |\mathbb{P}_1^\perp \beta_3| > \dots > r_1$$

Because $|\mathbb{P}_1^\perp \beta_i| = |\mathbb{P}_1^\perp (\beta_i + n\gamma_1)|$ for all n , we may assume, without loss of generality, that $|\mathbb{P}_1 \beta_i| \leq |\gamma_1|$ for every i . Because

$$\{ \mathbf{x} \in \mathbb{R}^d \mid |\mathbb{P}_1^\perp \mathbf{x}| \leq 2r_1, |\mathbb{P}_1 \mathbf{x}| \leq |\gamma_1| \}$$

is compact, the sequence has a limit point and hence has a Cauchy subsequence. In particular, there are β_i and β_j in the sequence, with $\beta_i \neq \beta_j$ with $|\beta_i - \beta_j| < \frac{r}{2}$. But this is impossible, because $\beta_i - \beta_j$ would be a nonzero element of \mathcal{P} with length smaller than r .

Claim 4. Let γ_2 be an element of $\mathcal{P} \setminus \mathcal{P}_1$ that minimizes $|\mathbb{P}_1^\perp \gamma|$ and set

$$\mathcal{P}_2 = \mathcal{P} \cap \{ x_1\gamma_1 + x_2\gamma_2 \mid x_1, x_2 \in \mathbb{R} \}$$

Then $\mathcal{P}_2 = \{ n_1\gamma_1 + n_2\gamma_2 \mid n_1, n_2 \in \mathbb{Z} \}$.

Proof of Claim 4: If $x_1\boldsymbol{\gamma}_1 + x_2\boldsymbol{\gamma}_2 \in \mathcal{P}$ with x_2 not an integer, then $\boldsymbol{\gamma}' = x_1\boldsymbol{\gamma}_1 + (x_2 - [x_2])\boldsymbol{\gamma}_2$ is an element of $\mathcal{P} \setminus \mathcal{P}_1$ with $|\mathbb{P}_1^\perp \boldsymbol{\gamma}'| = |x_2 - [x_2]| |\mathbb{P}_1^\perp \boldsymbol{\gamma}_2| < |\mathbb{P}_1^\perp \boldsymbol{\gamma}_2|$. So x_2 must be an integer. But then $(x_1\boldsymbol{\gamma}_1 + x_2\boldsymbol{\gamma}_2) - x_2\boldsymbol{\gamma}_2 = x_1\boldsymbol{\gamma}_1 \in \mathcal{P}$ and, by Claim 2, x_1 must be an integer as well.

If $\mathcal{P} = \mathcal{P}_2$, we have finished. Otherwise continue with

■

To exclude functions, as in Example L.2.c, that are “mixed periodic/non-periodic”, we shall assume that $d' = d$. Let $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d \in \mathbb{R}^d$ be d linearly independent vectors and set

$$\Gamma = \{ n_1\boldsymbol{\gamma}_1 + \dots + n_d\boldsymbol{\gamma}_d \mid n_1, \dots, n_d \in \mathbb{Z} \}$$

Γ is called the lattice generated by $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d$.

Problem L.1 The set of generators for a lattice are not uniquely determined. Let Γ be generated by d linearly independent vectors $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d \in \mathbb{R}^d$. Let Γ' be generated by d linearly independent vectors $\boldsymbol{\gamma}'_1, \dots, \boldsymbol{\gamma}'_d \in \mathbb{R}^d$. Prove that $\Gamma = \Gamma'$ if and only there is a $d \times d$ matrix A with integer matrix elements and $|\det A| = 1$ such that $\boldsymbol{\gamma}'_i = \sum_{j=1}^d A_{i,j}\boldsymbol{\gamma}_j$.

Problem L.2 Let $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d \in \mathbb{R}^d$ be d linearly independent vectors. Prove that there are two constants C and c , depending only on $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d$ such that

$$c|\mathbf{x}| \leq |x_1\boldsymbol{\gamma}_1 + \dots + x_d\boldsymbol{\gamma}_d| \leq C|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^d$.

We'll now find a bunch of functions that are periodic with respect to Γ . Consider $f(\mathbf{x}) = e^{i\mathbf{b} \cdot \mathbf{x}}$. This function has period $\boldsymbol{\gamma}$ if and only if $e^{i\mathbf{b} \cdot (\mathbf{x} + \boldsymbol{\gamma})} = e^{i\mathbf{b} \cdot \mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^d$. This is the case if and only if $e^{i\mathbf{b} \cdot \boldsymbol{\gamma}} = 1$ and this is the case if and only if $\mathbf{b} \cdot \boldsymbol{\gamma} \in 2\pi\mathbb{Z}$.

Definition L.4 Let Γ be a lattice in \mathbb{R}^d . The dual lattice for Γ is

$$\Gamma^\# = \{ \mathbf{b} \in \mathbb{R}^d \mid \mathbf{b} \cdot \boldsymbol{\gamma} \in 2\pi\mathbb{Z} \text{ for all } \boldsymbol{\gamma} \in \Gamma \}$$

Remark L.5 Let $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d \in \mathbb{R}^d$ be linearly independent and denote by Γ the lattice that they generate. A vector $\mathbf{b} \in \mathbb{R}^d$ is an element of $\Gamma^\#$ if and only if

$$\mathbf{b} \cdot \boldsymbol{\gamma}_j \in 2\pi\mathbb{Z} \quad \text{for all } 1 \leq j \leq d$$

Example L.6 Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard basis of \mathbb{R}^d . That is, \mathbf{e}_j has all components zero, except for the j^{th} , which is one. Choosing $\ell_1, \dots, \ell_d > 0$ and $\boldsymbol{\gamma}_j = \ell_j \mathbf{e}_j$,

$$\Gamma = \{ (n_1 \ell_1, \dots, n_d \ell_d) \mid n_1, \dots, n_d \in \mathbb{Z} \}$$

Then (x_1, \dots, x_d) is in $\Gamma^\#$ if and only if

$$(x_1, \dots, x_d) \cdot \boldsymbol{\gamma}_j = \ell_j x_j \in 2\pi\mathbb{Z} \iff x_j \in \frac{2\pi}{\ell_j}\mathbb{Z}$$

Thus

$$\Gamma^\# = \{ (n_1 \frac{2\pi}{\ell_1}, \dots, n_d \frac{2\pi}{\ell_d}) \mid n_1, \dots, n_d \in \mathbb{Z} \}$$

Example L.7 Let

$$\Gamma = \{ n(1, 0) + m(\pi, 1) \mid n, m \in \mathbb{Z} \}$$

Then

$$\Gamma^\# = \{ n(0, 2\pi) + m(2\pi, -2\pi^2) \mid n, m \in \mathbb{Z} \}$$

Since

$$[n'(1, 0) + m'(\pi, 1)] \cdot [n(0, 2\pi) + m(2\pi, -2\pi^2)] = 2\pi(n'm + m'n)$$

every vector of the form $n(0, 2\pi) + m(2\pi, -2\pi^2)$ with $m, n \in \mathbb{Z}$ is indeed in $\Gamma^\#$. To verify that only vectors of this form are in $\Gamma^\#$, let $\mathbf{z} = x(0, 2\pi) + y(2\pi, -2\pi^2)$ be any vector in \mathbb{R}^2 . (Certainly, $(0, 2\pi)$ and $(2\pi, -2\pi^2)$ form a basis for \mathbb{R}^2 .) For \mathbf{z} to be in $\Gamma^\#$ it is necessary that

$$\mathbf{z} \cdot (1, 0) = 2\pi y \in 2\pi\mathbb{Z}$$

$$\mathbf{z} \cdot (\pi, 1) = 2\pi x \in 2\pi\mathbb{Z}$$

which forces $x, y \in \mathbb{Z}$.

Problem L.3 Let Γ be generated by $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d \in \mathbb{R}^d$ (assumed linearly independent) and let

$$[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d] = \left\{ \sum_{j=1}^d t_j \boldsymbol{\gamma}_j \mid 0 \leq t_j \leq 1 \text{ for all } 1 \leq j \leq d \right\}$$

be the parallelepiped with the $\boldsymbol{\gamma}_j$'s as edges. Prove that if $\mathbf{b} \in \Gamma^\#$, then

$$\int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} e^{i\mathbf{b} \cdot \mathbf{x}} = \begin{cases} |[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]| & \text{if } \mathbf{b} = \mathbf{0} \\ 0 & \text{if } \mathbf{b} \neq \mathbf{0} \end{cases}$$

where $|[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]|$ is the volume of $[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]$. By Problem L.1, the volume $|[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]|$ is independent of the choice of generators. That is, if Γ is also generated by $\boldsymbol{\gamma}'_1, \dots, \boldsymbol{\gamma}'_d \in \mathbb{R}^d$, then $|[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]| = |[\boldsymbol{\gamma}'_1, \dots, \boldsymbol{\gamma}'_d]|$. Consequently, it is legitimate to define $|\Gamma| = |[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]|$.

Hence

$$\frac{1}{|\Gamma|} \int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} e^{i\mathbf{b} \cdot \mathbf{x}} = \begin{cases} 1 & \text{if } \mathbf{b} = \mathbf{0} \\ 0 & \text{if } \mathbf{b} \neq \mathbf{0} \end{cases}$$

Proposition L.8 *If $\gamma_1, \dots, \gamma_d \in \mathbb{R}^d$ are linearly independent and*

$$\Gamma = \{ n_1\gamma_1 + \dots + n_d\gamma_d \mid n_1, \dots, n_d \in \mathbb{Z} \}$$

then there exist d linearly independent vectors $\mathbf{b}_1, \dots, \mathbf{b}_d \in \mathbb{R}^d$ such that

$$\Gamma^\# = \{ n_1\mathbf{b}_1 + \dots + n_d\mathbf{b}_d \mid n_1, \dots, n_d \in \mathbb{Z} \}$$

Proof: For each $1 \leq i \leq d$

$$V_i = \{ x_1\gamma_1 + \dots + x_d\gamma_d \mid x_1, \dots, x_d \in \mathbb{R}, x_i = 0 \}$$

is a $d - 1$ dimensional subspace of \mathbb{R}^d . So V_i^\perp is a one dimensional subspace of \mathbb{R}^d . Let \mathbf{B}_i be any nonzero element of V_i^\perp and define

$$\mathbf{b}_i = \frac{2\pi}{\gamma_i \cdot \mathbf{B}_i} \mathbf{B}_i$$

Note that $\gamma_i \cdot \mathbf{B}_i$ cannot vanish because then γ_i would have to be in V_i , i.e. would have to be a linear combination of $\gamma_j, j \neq i$. Denote

$$B = \{ n_1\mathbf{b}_1 + \dots + n_d\mathbf{b}_d \mid n_1, \dots, n_d \in \mathbb{Z} \}$$

As

$$\mathbf{b}_i \cdot \gamma_j = \begin{cases} 2\pi & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

we have that $\mathbf{b}_i \in \Gamma^\#$ and hence $B \subset \Gamma^\#$.

If $x_1\mathbf{b}_1 + \dots + x_d\mathbf{b}_d = \mathbf{0}$ then $(x_1\mathbf{b}_1 + \dots + x_d\mathbf{b}_d) \cdot \gamma_j = 2\pi x_j = 0$ for every $1 \leq j \leq d$. So the \mathbf{b}_i 's are linearly independent and every vector in \mathbb{R}^d may be written in the form $x_1\mathbf{b}_1 + \dots + x_d\mathbf{b}_d$. If $x_1\mathbf{b}_1 + \dots + x_d\mathbf{b}_d \in \Gamma^\#$, then

$$(x_1\mathbf{b}_1 + \dots + x_d\mathbf{b}_d) \cdot \gamma_j = 2\pi x_j \in 2\pi\mathbb{Z}$$

so that $x_j \in \mathbb{Z}$ for every $1 \leq j \leq d$. Hence $\gamma^\# \subset B$. ■

From now on, we fix d linearly independent vectors $\gamma_1, \dots, \gamma_d \in \mathbb{R}^d$, set

$$\Gamma = \{ n_1\gamma_1 + \dots + n_d\gamma_d \mid n_1, \dots, n_d \in \mathbb{Z} \}$$

The set of all C^∞ functions on \mathbb{R}^d that are periodic with respect to Γ is denoted $C^\infty(\mathbb{R}^d/\Gamma)$. We have already observed that $f(\mathbf{x}) = e^{i\mathbf{b} \cdot \mathbf{x}}$ is in $C^\infty(\mathbb{R}^d/\Gamma)$ if and only in $\mathbf{b} \in \Gamma^\#$.

Remark L.9 Here is the story (at least in short form) behind the notation $C^\infty(\mathbb{R}^d/\Gamma)$. We have already observed that \mathbb{R}^d is a group (under addition) and that Γ is a subgroup of \mathbb{R}^d . As \mathbb{R}^d is abelian, all subgroups are normal and the set of equivalence classes under the equivalence relation

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} - \mathbf{y} \in \Gamma$$

is itself a group, denoted, as usual \mathbb{R}^d/Γ . Precisely, the equivalence class of $\mathbf{x} \in \mathbb{R}^d$ is $[\mathbf{x}] = \{ \mathbf{y} \in \mathbb{R}^d \mid \mathbf{x} \sim \mathbf{y} \} \subset \mathbb{R}^d$ and $\mathbb{R}^d/\Gamma = \{ [\mathbf{x}] \mid \mathbf{x} \in \mathbb{R}^d \}$. The group operation in \mathbb{R}^d/Γ is

$$[\mathbf{x}] + [\mathbf{y}] = [\mathbf{x} + \mathbf{y}]$$

As well as being a group, \mathbb{R}^d/Γ can also be turned into a smooth manifold, called a d -dimensional torus. If \mathcal{O} is any open subset of \mathbb{R}^d with the property that no two points of \mathcal{O} are equivalent under \sim , then the map

$$\begin{aligned} \xi_{\mathcal{O}} : \mathcal{O} &\rightarrow \mathbb{R}^d/\Gamma \\ \mathbf{x} &\mapsto [\mathbf{x}] \end{aligned}$$

is one-to-one. Its inverse is a coordinate map for \mathbb{R}^d/Γ . If Γ is generated by $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d$ and X is any point in \mathbb{R}^d , $\{ X + t_1\boldsymbol{\gamma}_1 + \dots + t_d\boldsymbol{\gamma}_d \mid 0 < t_j < 1 \text{ for all } 1 \leq j \leq d \}$ is one possible choice of \mathcal{O} . The notation $C^\infty(\mathbb{R}^d/\Gamma)$ designates the set of smooth (that is, C^∞) functions on the manifold \mathbb{R}^d/Γ .

Theorem L.10 (Fourier Series) *A function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is in $C^\infty(\mathbb{R}^d/\Gamma)$ if and only if*

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{|\Gamma|} \sum_{\mathbf{b} \in \Gamma^\#} \hat{f}_{\mathbf{b}} e^{i\mathbf{b} \cdot \mathbf{x}} \\ \text{with } \sum_{\mathbf{b} \in \Gamma^\#} |\mathbf{b}|^{2n} |\hat{f}_{\mathbf{b}}| &< \infty \quad \text{for all } n \in \mathbb{N} \end{aligned}$$

Furthermore, in this case,

$$\hat{f}_{\mathbf{b}} = \int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} e^{-i\mathbf{b} \cdot \mathbf{x}} f(\mathbf{x})$$

Proof of “if”: Suppose that we are given $\hat{f}_{\mathbf{b}}$, $\mathbf{b} \in \Gamma^\#$ obeying $\sum_{\mathbf{b} \in \Gamma^\#} |\mathbf{b}|^{2n} |\hat{f}_{\mathbf{b}}| < \infty$ for all $n \in \mathbb{N}$. In particular $\sum_{\mathbf{b} \in \Gamma^\#} |\hat{f}_{\mathbf{b}}| < \infty$ so the series $\frac{1}{|\Gamma|} \sum_{\mathbf{b} \in \Gamma^\#} \hat{f}_{\mathbf{b}} e^{i\mathbf{b} \cdot \mathbf{x}}$ converges absolutely and uniformly to some continuous function that is periodic with respect to Γ . Call the function $f(\mathbf{x})$. Furthermore for any $i_1, \dots, i_d \in \mathbb{N}$

$$\left| \left(\prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} \right) \hat{f}_{\mathbf{b}} e^{i\mathbf{b} \cdot \mathbf{x}} \right| = \left| \left(\prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_{\mathbf{b}} e^{i\mathbf{b} \cdot \mathbf{x}} \right| \leq |\mathbf{b}|^{\sum i_j} |\hat{f}_{\mathbf{b}}|$$

so the series $\frac{1}{|\Gamma|} \sum_{\mathbf{b} \in \Gamma^\#} \left(\prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} \right) \hat{f}_{\mathbf{b}} e^{i\mathbf{b} \cdot \mathbf{x}}$ also converges absolutely and uniformly. This implies that $f(\mathbf{x})$ is C^∞ . Furthermore

$$\begin{aligned}
\int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} e^{-i\mathbf{b} \cdot \mathbf{x}} f(\mathbf{x}) &= \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} e^{-i\mathbf{b} \cdot \mathbf{x}} \left[\frac{1}{|\Gamma|} \sum_{\mathbf{c} \in \Gamma^\#} \hat{f}_{\mathbf{c}} e^{i\mathbf{c} \cdot \mathbf{x}} \right] \\
&= \frac{1}{|\Gamma|} \sum_{\mathbf{c} \in \Gamma^\#} \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} e^{i(\mathbf{c} - \mathbf{b}) \cdot \mathbf{x}} \hat{f}_{\mathbf{c}} \\
&= \frac{1}{|\Gamma|} \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} \hat{f}_{\mathbf{b}} + \frac{1}{|\Gamma|} \sum_{\substack{\mathbf{c} \in \Gamma^\# \\ \mathbf{c} \neq \mathbf{b}}} \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} e^{i(\mathbf{c} - \mathbf{b}) \cdot \mathbf{x}} \hat{f}_{\mathbf{c}} \\
&= \hat{f}_{\mathbf{b}}
\end{aligned}$$

by Problem L.3.

Proof of “only if”: Now suppose that we are given $f \in C^\infty(\mathbb{R}^d/\Gamma)$. Define

$$\hat{f}_{\mathbf{b}} = \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} e^{-i\mathbf{b} \cdot \mathbf{x}} f(\mathbf{x})$$

Then for any $i_1, \dots, i_d \in \mathbb{N}$

$$\begin{aligned}
\left| \left(\prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_{\mathbf{b}} \right| &= \left| \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} \left(\prod_{j=1}^d b_{i_j}^{i_j} \right) e^{-i\mathbf{b} \cdot \mathbf{x}} f(\mathbf{x}) \right| \\
&= \left| \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} \left(\prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} e^{-i\mathbf{b} \cdot \mathbf{x}} \right) f(\mathbf{x}) \right| \\
&= \left| \int_{[\gamma_1, \dots, \gamma_d]} d^d \mathbf{x} e^{-i\mathbf{b} \cdot \mathbf{x}} \left(\prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} f(\mathbf{x}) \right) \right| \\
&\leq |\Gamma| \sup_{\mathbf{x}} \left(\prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} f(\mathbf{x}) \right) < \infty
\end{aligned}$$

so that, by Problem L.2,

$$\begin{aligned}
\sum_{\mathbf{b} \in \Gamma^\#} \left| \left(\prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_{\mathbf{b}} \right| &= \sum_{\mathbf{b} \in \Gamma^\#} \frac{1 + |\mathbf{b}|^{d+1}}{1 + |\mathbf{b}|^{d+1}} \left| \left(\prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_{\mathbf{b}} \right| \\
&\leq \left[\sup_{\mathbf{b} \in \Gamma^\#} (1 + |\mathbf{b}|^{d+1}) \left| \left(\prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_{\mathbf{b}} \right| \right] \sum_{\mathbf{b} \in \Gamma^\#} \frac{1}{1 + |\mathbf{b}|^{d+1}} \\
&\leq \left[\sup_{\mathbf{b} \in \Gamma^\#} (1 + |\mathbf{b}|^{d+1}) \left| \left(\prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_{\mathbf{b}} \right| \right] \sum_{\mathbf{n} \in \mathbb{Z}^d} \frac{1}{1 + (c|\mathbf{n}|)^{d+1}} < \infty
\end{aligned}$$

Hence, by the “only if” part of this Theorem, that we have already proven,

$$g(\mathbf{x}) = \frac{1}{|\Gamma|} \sum_{\mathbf{b} \in \Gamma^\#} \hat{f}_{\mathbf{b}} e^{i\mathbf{b} \cdot \mathbf{x}}$$

is a C^∞ function and

$$\hat{f}_{\mathbf{b}} = \int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} e^{-i\mathbf{b} \cdot \mathbf{x}} g(\mathbf{x}) \quad (\text{L.1})$$

We just have to show that $g(\mathbf{x}) = f(\mathbf{x})$.

Here is one proof that $g(\mathbf{x}) = f(\mathbf{x})$. By (L.1)

$$\int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} e^{-i\mathbf{b} \cdot \mathbf{x}} [g(\mathbf{x}) - f(\mathbf{x})] = 0$$

for all $\mathbf{b} \in \Gamma^\#$. Consequently,

$$\int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} \varphi(\mathbf{x}) [g(\mathbf{x}) - f(\mathbf{x})] = 0$$

for any function $\varphi \in \mathcal{P}(\Gamma^\#)$ where $\mathcal{P}(\Gamma^\#)$ is the set of all functions that are finite linear combinations of the $e^{-i\mathbf{b} \cdot \mathbf{x}}$'s with $\mathbf{b} \in \Gamma^\#$. Consequently,

$$\int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} \varphi(\mathbf{x}) [g(\mathbf{x}) - f(\mathbf{x})] = 0$$

for any function $\varphi \in \overline{\mathcal{P}(\Gamma^\#)}$ where $\overline{\mathcal{P}(\Gamma^\#)}$ is the set of all functions that are uniform limits of sequences of functions in $\mathcal{P}(\Gamma^\#)$. But by the Stone–Weierstrass Theorem [Walter Rudin, Principles of Mathematical Analysis, Theorem 7.33], $\overline{\mathcal{P}(\Gamma^\#)}$ is the set of all continuous functions that are periodic with respect to Γ . In particular, the complex conjugate of $g(\mathbf{x}) - f(\mathbf{x})$ is in $\overline{\mathcal{P}(\Gamma^\#)}$. Hence

$$\int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} |g(\mathbf{x}) - f(\mathbf{x})|^2 = 0$$

so that $g(\mathbf{x}) = f(\mathbf{x})$ for all \mathbf{x} .

One may also build Problem L.5, below, into a second proof that $g(\mathbf{x}) = f(\mathbf{x})$. Just make a change of variables so that Γ is replaced by $2\pi\mathbb{Z}^d$ and apply Problem L.5.b, once in each dimension. ■

Problem L.4 Let Γ be generated by $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d \in \mathbb{R}^d$ (assumed linearly independent) and also by $\boldsymbol{\gamma}'_1, \dots, \boldsymbol{\gamma}'_d \in \mathbb{R}^d$ (also assumed linearly independent). Recall that

$$[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d] = \left\{ \sum_{j=1}^d t_j \boldsymbol{\gamma}_j \mid 0 \leq t_j \leq 1 \text{ for all } 1 \leq j \leq d \right\}$$

is the parallelepiped with the $\boldsymbol{\gamma}_j$'s as edges. Let, for $\mathbf{y} \in \mathbb{R}^d$,

$$\mathbf{y} + [\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d] = \left\{ \mathbf{y} + \sum_{j=1}^d t_j \boldsymbol{\gamma}_j \mid 0 \leq t_j \leq 1 \text{ for all } 1 \leq j \leq d \right\}$$

be the translate of $[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]$ by \mathbf{y} . Let $f(\mathbf{x})$ be periodic with respect to Γ . Prove that

$$\int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} f(\mathbf{x}) = \int_{\mathbf{y} + [\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} f(\mathbf{x}) = \int_{[\boldsymbol{\gamma}'_1, \dots, \boldsymbol{\gamma}'_d]} d^d \mathbf{x} f(\mathbf{x})$$

We denote

$$\int_{\mathbb{R}^d/\Gamma} d^d \mathbf{x} f(\mathbf{x}) = \int_{[\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d]} d^d \mathbf{x} f(\mathbf{x})$$

Problem L.5 Let $f \in C^1(\mathbb{R})$ be periodic of period 2π . Set

$$c_n = \int_0^{2\pi} e^{-inx} f(x) dx$$

and

$$(S_M f)(\theta) = \frac{1}{2\pi} \sum_{n=-M}^M c_n e^{in\theta}$$

- a) Prove that $S_M f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta + x) \frac{\sin((M+1/2)x)}{\sin(x/2)} dx$.
- b) Prove that $S_M f(\theta)$ converges to $f(\theta)$ as $M \rightarrow \infty$.