

Families of Commuting Normal Matrices

Definition M.1 (Notation)

i) $\mathbb{C}^n = \{ \mathbf{v} = (v_1, \dots, v_n) \mid v_i \in \mathbb{C} \text{ for all } 1 \leq i \leq n \}$

ii) If $\lambda \in \mathbb{C}$ and $\mathbf{v} = (v_1, \dots, v_n)$, $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{C}^n$, then

$$\begin{aligned} \lambda \mathbf{v} &= (\lambda v_1, \dots, \lambda v_n) \in \mathbb{C}^n \\ \mathbf{v} + \mathbf{w} &= (v_1 + w_1, \dots, v_n + w_n) \in \mathbb{C}^n \\ \langle \mathbf{v}, \mathbf{w} \rangle &= \sum_{j=1}^n \bar{v}_j w_j \in \mathbb{C} \end{aligned}$$

The $\bar{}$ means complex conjugate.

iii) Two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ are said to be orthogonal (or perpendicular, denoted $\mathbf{v} \perp \mathbf{w}$) if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

iv) If $\mathbf{v} \in \mathbb{C}^n$ and A is the $m \times n$ matrix whose (i, j) matrix element is $A_{i,j}$, then $A\mathbf{v}$ is the vector in \mathbb{C}^m with

$$(A\mathbf{v})_i = \sum_{j=1}^n A_{i,j} v_j \quad \text{for all } 1 \leq i \leq m$$

v) A linear subspace V of \mathbb{C}^n is a subset of \mathbb{C}^n that is closed under addition and scalar multiplication. That is, if $\lambda \in \mathbb{C}$ and $\mathbf{v}, \mathbf{w} \in V$, then $\lambda \mathbf{v}$, $\mathbf{v} + \mathbf{w} \in V$.

vi) If V is a subset of \mathbb{C}^n , then its orthogonal complement is

$$V^\perp = \{ \mathbf{v} \in \mathbb{C}^n \mid \mathbf{v} \perp \mathbf{w} \text{ for all } \mathbf{w} \in V \}$$

Problem M.1 Let $V \subset \mathbb{C}^n$. Prove that V^\perp is a linear subspace of \mathbb{C}^n .

Lemma M.2 Let V be a linear subspace of \mathbb{C}^n of dimension at least one. Let A be an $n \times n$ matrix that maps V into V . Then A has an eigenvector in V .

Proof: Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be a basis for V . As A maps V into itself, there exist numbers $a_{i,j}$, $1 \leq i, j \leq d$ such that

$$A\mathbf{e}_j = \sum_{i=1}^d a_{i,j} \mathbf{e}_i \quad \text{for all } 1 \leq j \leq d$$

Consequently, A maps the vector $\mathbf{w} = \sum_{j=1}^d x_j \mathbf{e}_j \in V$ to

$$A\mathbf{w} = \sum_{i,j=1}^d a_{i,j} x_j \mathbf{e}_i$$

so that \mathbf{w} is an eigenvector of A of eigenvalue λ if and only if (1) not all of the x_i 's are zero and (2)

$$\begin{aligned} \sum_{i,j=1}^d a_{i,j} x_j \mathbf{e}_i = \lambda \sum_{i=1}^d x_i \mathbf{e}_i &\iff \sum_{j=1}^d a_{i,j} x_j = \lambda x_i && \text{for all } 1 \leq i \leq d \\ &\iff \sum_{j=1}^d (a_{i,j} - \lambda \delta_{i,j}) x_j = 0 && \text{for all } 1 \leq i \leq d \end{aligned}$$

For any given λ , the linear system of equations " $\sum_{j=1}^d (a_{i,j} - \lambda \delta_{i,j}) x_j = 0$ for all $1 \leq i \leq d$ " has a nontrivial solution (x_1, \dots, x_d) if and only if the $d \times d$ matrix $[a_{i,j} - \lambda \delta_{i,j}]_{1 \leq i,j \leq d}$ fails to be invertible and this, in turn, is the case if and only if $\det [a_{i,j} - \lambda \delta_{i,j}] = 0$. But $\det [a_{i,j} - \lambda \delta_{i,j}] = 0$ is a polynomial of degree d in λ and so always vanishes for at least one value of λ . ■

Definition M.3 (Commuting) Two $n \times n$ matrices A and B are said to commute if $AB = BA$.

Lemma M.4 *Let $n \geq 1$ be an integer, V be a linear subspace of \mathbb{C}^n of dimension at least one and let \mathcal{F} be a nonempty set of $n \times n$ mutually commuting matrices that map V into V . That is, $A, B \in \mathcal{F} \Rightarrow AB = BA$ and $A \in \mathcal{F}, \mathbf{w} \in V \Rightarrow A\mathbf{w} \in V$. Then there exists a nonzero vector $\mathbf{v} \in V$ that is an eigenvector for every matrix in \mathcal{F} .*

Proof: We shall show that

"There is a linear subspace W of V of dimension at least one, such that each $A \in \mathcal{F}$ is a multiple of the identity matrix when restricted to W ."

This suffices to prove the lemma. The proof will be by induction on the dimension d of V . If $d = 1$, we may take $W = V$, since the restriction of any matrix to a one dimensional vector space is a multiple of the identity.

Suppose that the claim has been proven for all dimensions strictly less than d . If every $A \in \mathcal{F}$ is a multiple of the identity, when restricted to V , we may take $W = V$ and we are done. If not, pick any $A \in \mathcal{F}$ that is not a multiple of the identity when restricted to V .

By Lemma M.2, it has at least one eigenvector $\mathbf{v} \in V$. Let λ be the corresponding eigenvalue and set

$$V' = V \cap \{ \mathbf{w} \in \mathbb{C}^n \mid A\mathbf{w} = \lambda\mathbf{w} \}$$

Then V' is a linear subspace of V of dimension strictly less than d (since A , restricted to V , is not $\lambda\mathbb{1}$). We claim that every $B \in \mathcal{F}$ maps V' into V' . To see this, let $B \in \mathcal{F}$ and $\mathbf{w} \in V'$ and set $\mathbf{w}' = B\mathbf{w}$. We wish to show that $\mathbf{w}' \in V'$. But

$$\begin{aligned} A\mathbf{w}' &= AB\mathbf{w} = BA\mathbf{w} && (A \text{ and } B \text{ commute}) \\ &= B\lambda\mathbf{w} && (\text{Definition of } V') \\ &= \lambda B\mathbf{w} = \lambda\mathbf{w}' \end{aligned}$$

so \mathbf{w}' is indeed in V' . We have verified that V' has dimension at least one and strictly smaller than d and that every $B \in \mathcal{F}$ maps V' into V' . So we may apply the inductive hypothesis with V replaced by V' . ■

Definition M.5 (Adjoint) The adjoint of the $r \times c$ matrix A is the $c \times r$ matrix

$$A_{i,j}^* = \overline{A_{j,i}}$$

Problem M.2 Let A and B be any $n \times n$ matrices. Prove that $B = A^*$ if and only if $\langle B\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A\mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$.

Problem M.3 Let A be any $n \times n$ matrix. Let V be any linear subspace of \mathbb{C}^n and V^\perp its orthogonal complement. Prove that if $AV \subset V$ (i.e. $\mathbf{w} \in V \Rightarrow A\mathbf{w} \in V$), then $A^*V^\perp \subset V^\perp$.

Definition M.6 (Normal, Self-Adjoint, Unitary)

- i) An $n \times n$ matrix A is normal if $AA^* = A^*A$. That is, if A commutes with its adjoint.
- ii) An $n \times n$ matrix A is self-adjoint if $A = A^*$.
- iii) An $n \times n$ matrix U is unitary if $UU^* = \mathbb{1}$. Here $\mathbb{1}$ is the $n \times n$ identity matrix. Its (i, j) matrix element is one if $i = j$ and zero otherwise.

Problem M.4 Let A be a normal matrix. Let λ be an eigenvalue of A and V the eigenspace of A of eigenvalue λ . Prove that V is the eigenspace of A^* of eigenvalue $\bar{\lambda}$.

Problem M.5 Let A be a normal matrix. Let \mathbf{v} and \mathbf{w} be eigenvectors of A with different eigenvalues. Prove that $\mathbf{v} \perp \mathbf{w}$.

Problem M.6 Let A be a self-adjoint matrix. Prove that

- a) A is normal
- b) Every eigenvalue of A is real.

Problem M.7 Let U be a unitary matrix. Prove that

- a) U is normal
- b) Every eigenvalue λ of U obeys $|\lambda| = 1$, i.e. is of modulus one.

Theorem M.7 Let $n \geq 1$ be an integer. Let \mathcal{F} be a nonempty set of $n \times n$ mutually commuting normal matrices. That is, $A, B \in \mathcal{F} \Rightarrow AB = BA$ and $A \in \mathcal{F} \Rightarrow AA^* = A^*A$. Then there exists an orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of \mathbb{C}^n such that \mathbf{e}_j is an eigenvector of A for every $A \in \mathcal{F}$ and $1 \leq j \leq n$.

Proof: By Lemma M.4, with $V = \mathbb{C}^n$, there exists a nonzero vector \mathbf{v}_1 that is an eigenvector for every $A \in \mathcal{F}$. Set $\mathbf{e}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$ and $V_1 = \{ \lambda \mathbf{e}_1 \mid \lambda \in \mathbb{C} \}$. By Problem M.4, \mathbf{e}_1 is also an eigenvector of A^* for every $A \in \mathcal{F}$, so $A^*V_1 \subset V_1$ for all $A \in \mathcal{F}$. By Problem M.3, $AV_1^\perp \subset V_1^\perp$ for all $A \in \mathcal{F}$.

By Lemma M.4, with $V = V_1^\perp$, there exists a nonzero vector $\mathbf{v}_2 \in V_1^\perp$ that is an eigenvector for every $A \in \mathcal{F}$. Choose $\mathbf{e}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$. As $\mathbf{e}_2 \in V_1^\perp$, \mathbf{e}_2 is orthogonal to \mathbf{e}_1 . Define $V_2 = \{ \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \mid \lambda_1, \lambda_2 \in \mathbb{C} \}$. By Problem M.4, \mathbf{e}_2 is also an eigenvector of A^* for every $A \in \mathcal{F}$, so $A^*V_2 \subset V_2$ for all $A \in \mathcal{F}$. By Problem M.3, $AV_2^\perp \subset V_2^\perp$ for all $A \in \mathcal{F}$.

By Lemma M.4, with $V = V_2^\perp$, there exists a nonzero vector $\mathbf{v}_3 \in V_2^\perp$ that is an eigenvector for every $A \in \mathcal{F}$. Choose $\mathbf{e}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$. As $\mathbf{e}_3 \in V_2^\perp$, \mathbf{e}_3 is orthogonal to both \mathbf{e}_1 and \mathbf{e}_2 . And so on. ■