The Principle of Uniform Boundedness, and Friends

In these notes, unless otherwise stated, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is linear and has domain $\mathcal{X}$.

Theorem 1

(a) T is bounded if and only if

$$T^{-1}\big\{\ \mathbf{y} \in \mathcal{Y}\ \big|\ \|\mathbf{y}\|_{\mathcal{Y}} \le 1\ \big\} = \big\{\ \mathbf{x} \in \mathcal{X}\ \big|\ \|T\mathbf{x}\|_{\mathcal{Y}} \le 1\ \big\}$$

has nonempty interior. ($\mathcal{X}, \mathcal{Y}$ need not be complete.)

(b) **Principle of Uniform Boundedness:** *Let $\mathcal{F} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$.*

$$\begin{array}{ll} \textit{If, for each } \mathbf{x} \in X, & \big\{\ \|T\mathbf{x}\|\ \big|\ T \in \mathcal{F}\ \big\} \textit{is bounded,} \\ \textit{then} & \big\{\ \|T\|\ \big|\ T \in \mathcal{F}\ \big\} \textit{is bounded,} \end{array}$$

($\mathcal{Y}$ need not be complete.)

(c) If $B : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ is bilinear and continuous in each variable separately (i.e. $B(\mathbf{x}, \mathbf{y})$ is continuous in $\mathbf{x}$ for each fixed $\mathbf{y}$ and vice versa), then $B(\mathbf{x}, \mathbf{y})$ is jointly continuous (i.e. if $\lim_{n\to\infty} \mathbf{x}_n = 0$ and $\lim_{n\to\infty} \mathbf{y}_n = 0$, then $\lim_{n\to\infty} B(\mathbf{x}_n, \mathbf{y}_n) = 0$.

(d) **Open Mapping Theorem:** *If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is surjective (i.e. onto) and if $\mathcal{O}$ is an open subset of $\mathcal{X}$, then $T\mathcal{O} = \big\{\ T\mathbf{x}\ \big|\ \mathbf{x} \in \mathcal{O}\ \big\}$ is an open subset of $\mathcal{Y}$.*

(e) **Inverse Mapping Theorem:** *If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is bijective (i.e. 1–1 and onto), then T^{-1} is bounded.*

(f) **Closed Graph Theorem:** *The graph of T is defined to be*

$$\Gamma(T) = \big\{\ (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}\ \big|\ \mathbf{y} = T\mathbf{x}\ \big\}$$

Then

$$T \textit{ is bounded} \iff \Gamma(T) \textit{ is closed}$$

(g) **Hellinger–Toeplitz Theorem:** *Let T be an everywhere defined linear operator on $\mathcal{H}$ that obeys $\langle \mathbf{x}, T\mathbf{y} \rangle = \langle T\mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Then T is bounded.*

Remark 2 The linearity of T is crucial for Theorem 1. For part (c), for example, the function $T : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by

$$T(x,y) = \begin{cases} \frac{|xy|}{|x|^2+|y|^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous in x and y separately, but not in (x,y) jointly. (The limit as $(x,y) \to (0,0)$ with $x = y$ is $\frac{1}{2}$, while the limit as $(x,y) \to (0,0)$ with $x = 0$ is 0.)

Proof of Theorem 1.a:

$\Leftarrow$: By hypothesis, there are $\varepsilon > 0$ and $\mathbf{x}_0 \in \mathcal{X}$ such that

$$\|T(\mathbf{x}+\mathbf{x}_0)\|_{\mathcal{Y}} \le 1 \qquad \text{for all } \mathbf{x} \in \mathcal{X} \text{ with } \|\mathbf{x}\|_{\mathcal{X}} < \varepsilon$$

Since T is linear, if $\mathbf{x} \in \mathcal{X}$ with $\|\mathbf{x}\|_{\mathcal{X}} < \varepsilon$, then

$$\|T\mathbf{x}\|_{\mathcal{Y}} = \|T(\mathbf{x}+\mathbf{x}_0) - T\mathbf{x}_0\|_{\mathcal{Y}} \le \|T(\mathbf{x}+\mathbf{x}_0)\|_{\mathcal{Y}} + \|T\mathbf{x}_0\|_{\mathcal{Y}} \le 1 + \|T\mathbf{x}_0\|_{\mathcal{Y}}$$

So, for any $0 \neq \mathbf{x} \in \mathcal{X}$ and any $0 < \varepsilon' < \varepsilon$,

$$\|T\mathbf{x}\|_{\mathcal{Y}} = \tfrac{\|\mathbf{x}\|_{\mathcal{X}}}{\varepsilon'} \big\|T\big(\varepsilon' \tfrac{\mathbf{x}}{\|\mathbf{x}\|_{\mathcal{X}}}\big)\big\|_{\mathcal{Y}} \le \tfrac{1}{\varepsilon'}\big[1 + \|T\mathbf{x}_0\|_{\mathcal{Y}}\big]\, \|\mathbf{x}\|_{\mathcal{X}}$$

Hence $\|T\| \le \frac{1}{\varepsilon}\big[1 + \|T\mathbf{x}_0\|_{\mathcal{Y}}\big]$.

$\Rightarrow$: By hypothesis $\|T\|$ is finite. Hence

$$\mathbf{x} \in \mathcal{X},\ \|\mathbf{x}\|_{\mathcal{X}} < \tfrac{1}{\|T\|} \implies \|T\mathbf{x}\|_{\mathcal{Y}} \le \|T\|\,\|\mathbf{x}\|_{\mathcal{X}} < 1$$

So $\big\{\ \mathbf{x} \in \mathcal{X}\ \big|\ \|\mathbf{x}\|_{\mathcal{X}} < \frac{1}{\|T\|}\ \big\} \subset \big\{\ \mathbf{x} \in \mathcal{X}\ \big|\ \|T\mathbf{x}\|_{\mathcal{Y}} \le 1\ \big\}$ and the latter has nonempty interior. ■

Definition 3 (Nowhere Dense) A subset S of a metric space $\mathcal{M}$ is said to be *nowhere dense* if its closure $\overline{S}$ has empty interior. Equivalently, S is nowhere dense if it is not dense in any open ball of $\mathcal{M}$.

Theorem 4 (Baire Category Theorem) *Let $\mathcal{M}$ be a nonempty complete metric space. Then $\mathcal{M}$ cannot be written as the union of a countable number of nowhere dense subsets.*

Proof: The proof is by contradiction. Suppose that $\mathcal{M} = \bigcup_{n=1}^{\infty} S_n$, with each S_n nowhere dense. We shall construct a Cauchy sequence $\big\{x_n\big\}_{n\in\mathbb{N}}$ whose limit $x \notin \bigcup_{n=1}^{\infty} S_n$.

- Since S_1 is nowhere dense, $\overline{S_1} \neq \mathcal{M}$. So there must exist some $x_1 \in \mathcal{M}$ and some $0 < r_1 < \frac{1}{2}$ such that $B_{r_1}(x_1) \cap S_1 = \emptyset$. Here, as usual, $B_{r_1}(x_1)$ denotes the open ball of radius r_1 centred on x_1.
- Since S_2 is nowhere dense, $\overline{S_2}$ may not contain all of $B_{r_1}(x_1)$. So there must exist some $x_2 \in B_{r_1}(x_1)$ and some $0 < r_2 < \frac{1}{4}$ and $B_{r_2}(x_2) \cap S_2 = \emptyset$. Possibly shrinking r_2, we may assume that $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1)$.
- Continue in this way to construct inductively, for each $n \in \mathbb{N}$, an $x_{n+1} \in B_{r_n}(x_n)$ and a $0 < r_{n+1} < \frac{1}{2^{n+1}}$ such that $B_{r_{n+1}}(x_{n+1}) \cap S_{n+1} = \emptyset$ and $\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n)$.

Write, for each $n \in \mathbb{N}$, $B_n = B_{r_n}(x_n)$. We have now constructed

$$\begin{array}{rcl} x_1 \in & B_1 & \text{with } B_1 \cap S_1 = \emptyset \\ & \cup & \\ & \overline{B_2} & \\ & \cup & \\ x_2 \in & B_2 & \text{with } B_2 \cap S_2 = \emptyset \\ & \cup & \\ & \overline{B_3} & \\ & \cup & \\ x_3 \in & B_3 & \text{with } B_3 \cap S_3 = \emptyset \\ & \vdots & \end{array}$$

For each $n \in \mathbb{N}$, $x_{n+1} \in B_{r_n}(x_n)$. In particular, $d\big(x_n, x_{n+1}\big) < r_n \le \frac{1}{2^n}$. Consequently $\big\{x_n\big\}_{n\in\mathbb{N}}$ is Cauchy. As $\mathcal{M}$ is complete, it has a limit, x. Since $x_n \in B_m$ for all $n \ge m$, we have $x \in \overline{B_m}$ for all $m \ge 2$ and hence $x \in B_\ell$ for all $\ell \in \mathbb{N}$. So $x \notin S_\ell$ for all $\ell \in \mathbb{N}$. ■

Proof of Theorem 1.b: Set $\mathcal{M} = \mathcal{X}$ and, for each $n \in \mathbb{N}$,

$$S_n = \big\{ \mathbf{x} \in \mathcal{X} \bigm| \|T\mathbf{x}\|_{\mathcal{Y}} \le n \text{ for all } T \in \mathcal{F} \big\} = \bigcap_{T\in\mathcal{F}} T^{-1}\big\{ \mathbf{y} \in \mathcal{Y} \bigm| \|\mathbf{y}\|_{\mathcal{Y}} \le n \big\}$$

By hypothesis $\mathcal{M} = \bigcup_{n=1}^\infty S_n$. Since each $T \in \mathcal{F}$ is continuous, each S_n, $n \in \mathbb{N}$ is closed. By the Baire category theorem, there is an $n_0 \in \mathbb{N}$ such that S_{n_0} has nonempty interior. Just as in the proof of of Theorem 1.a, there is a $\varepsilon > 0$ and an $\mathbf{x}_0 \in \mathcal{X}$ such that

$$\|T\| \le \tfrac{1}{\varepsilon}\big[n_0 + \|T\mathbf{x}_0\|_{\mathcal{Y}}\big] \le \tfrac{1}{\varepsilon}\big[n_0 + \sup_{T'\in\mathcal{F}} \|T'\mathbf{x}_0\|_{\mathcal{Y}}\big] < \infty$$

for all $T \in \mathcal{F}$. ■

Proof of Theorem 1.c: Let $\lim_{n\to\infty} \mathbf{x}_n = 0$ and $\lim_{n\to\infty} \mathbf{y}_n = 0$. For each $n \in \mathbb{N}$, define $T_n : \mathcal{Y} \to \mathbb{C}$ by $T_n\mathbf{y} = B(\mathbf{x}_n, \mathbf{y})$. Since T_n is continuous (recall that, for each fixed $\mathbf{x} \in \mathcal{X}$, $B(\mathbf{x}, \mathbf{y})$ is continuous in $\mathbf{y}$) and linear, it is bounded. For each fixed $\mathbf{y} \in \mathcal{Y}$, $\lim_{n\to\infty} T_n\mathbf{y} = 0$, since $B(\mathbf{x}, \mathbf{y})$ is continuous in $\mathbf{x}$ when $\mathbf{y}$ is held fixed. Thus, for each fixed $\mathbf{y} \in \mathcal{Y}$, $\big\{ T_n\mathbf{y} \bigm| n \in \mathbb{N} \big\}$ is bounded. Hence, by the Principle of Uniform Boundedness, $\tau = \sup\big\{ \|T_n\| \bigm| n \in \mathbb{N} \big\} < \infty$ so that,

$$\big|B(\mathbf{x}_n, \mathbf{y}_n)\big| = \|T_n\mathbf{y}_n\| \le \|T_n\|\,\|\mathbf{y}_n\| \le \tau\|\mathbf{y}_n\| \to 0 \quad \text{as } n \to \infty$$

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Proof of Theorem 1.d:

Step 1: I claim that it suffices to prove that there is an $\varepsilon > 0$ such that

$$B_\varepsilon(\mathbf{0}) \subset TB_2(\mathbf{0}) = \big\{ T\mathbf{x} \bigm| \|\mathbf{x}\|_{\mathcal{X}} < 2 \big\} \tag{1}$$

Because then,

- for every $r > 0$, $B_{\frac{1}{2}r\varepsilon}(\mathbf{0}) \subset TB_r(\mathbf{0})$, by scaling, so that $\mathbf{0}$ is in the interior of $TB_r(\mathbf{0})$
- and for every $\mathbf{x} \in \mathcal{X}$ and every $r > 0$, $B_{\frac{1}{2}r\varepsilon}(T\mathbf{x}) \subset TB_r(\mathbf{x})$, by translating, so that $T\mathbf{x}$ is in the interior of $TB_r(\mathbf{x})$
- so that, if we choose, for each $\mathbf{x} \in \mathcal{O}$, $r_{\mathbf{x}} > 0$ sufficiently small that $B_{r_{\mathbf{x}}}(\mathbf{x}) \subset \mathcal{O}$,

$$T\mathcal{O} = \bigcup_{\mathbf{x}\in\mathcal{O}} \{T\mathbf{x}\} \subset \bigcup_{\mathbf{x}\in\mathcal{O}} \big(TB_{r_{\mathbf{x}}}(\mathbf{x})\big)^\circ \subset \bigcup_{\mathbf{x}\in\mathcal{O}} \big(T\mathcal{O}\big)^\circ = \big(T\mathcal{O}\big)^\circ$$

Step 2: We now prove that there is an $\varepsilon > 0$ obeying (1).

$$\begin{aligned} T \text{ surjective} \implies& \mathcal{Y} = \bigcup_{n\in\mathbb{N}} TB_n(\mathbf{0}) \\ \implies& \overline{TB_n(\mathbf{0})} \text{ has a nonempty interior for some } n \in \mathbb{N} \\ & \qquad\qquad \text{(by the Baire Category Theorem)} \\ \implies& B_\varepsilon(\mathbf{0}) \subset \overline{TB_1(\mathbf{0})} \text{ for some } \varepsilon > 0 \\ & \qquad\qquad \text{(by scaling and translating)} \end{aligned}$$

So it suffices to prove that $\overline{TB_1(\mathbf{0})} \subset TB_2(\mathbf{0})$. So let $\mathbf{y} \in \overline{TB_1(\mathbf{0})}$. Then

- $\exists\ \mathbf{x}_1 \in B_1(\mathbf{0})$ such that $\mathbf{y} - T\mathbf{x}_1 \in B_{\frac{\varepsilon}{2}}(\mathbf{0}) \subset \overline{TB_{\frac{1}{2}}(\mathbf{0})}$ and
- $\exists\ \mathbf{x}_2 \in B_{\frac{1}{2}}(\mathbf{0})$ such that $\mathbf{y} - T\mathbf{x}_1 - T\mathbf{x}_2 \in B_{\frac{\varepsilon}{4}}(\mathbf{0}) \subset \overline{TB_{\frac{1}{4}}(\mathbf{0})}$
- and so on

Set $\mathbf{x} = \sum_{n=1}^\infty \mathbf{x}_n$. Then $\|\mathbf{x}\|_{\mathcal{X}} \le \sum_{n=1}^\infty \|\mathbf{x}_n\|_{\mathcal{X}} < \sum_{n=1}^\infty \frac{1}{2^{n-1}} = 2$ so that $\mathbf{x} \in B_2(\mathbf{0})$ and $\mathbf{y} = \sum_{n=1}^\infty T\mathbf{x}_n = T\mathbf{x} \in TB_2(\mathbf{0})$. ■

Proof of Theorem 1.e: Since T is linear and bijective, it has a linear inverse, T^{-1}, defined on all of $\mathcal{Y}$. If $\mathcal{O}$ is any open subset of $\mathcal{X}$, then

$$\big(T^{-1}\big)^{-1}(\mathcal{O}) = T(\mathcal{O})$$

is open, by part (d), the open mapping theorem. So T^{-1} is continuous, by definition. ■

Proof of Theorem 1.f:

$\Rightarrow$: By hypothesis T is bounded. Suppose that the sequence $\big\{(\mathbf{x}_n, \mathbf{y}_n)\big\}_{n\in\mathbb{N}} \subset \Gamma(T)$ converges to $(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$. That is, assume that $\mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n$ and $\mathbf{y} = \lim_{n\to\infty} \mathbf{y}_n$. Then

$$\mathbf{y} = \lim_{n\to\infty} \mathbf{y}_n = \lim_{n\to\infty} T\mathbf{x}_n = T \lim_{n\to\infty} \mathbf{x}_n = T\mathbf{x}$$

Thus $(\mathbf{x}, \mathbf{y}) \in \Gamma(T)$ and $\Gamma(T)$ is closed.

$\Leftarrow$: By hypothesis $\Gamma(T)$ is closed. As a result, $\Gamma(T)$ is a Banach space with the norm $\big\|(\mathbf{x}, T\mathbf{x})\big\|_{\Gamma(T)} = \sqrt{\|\mathbf{x}\|_{\mathcal{X}}^2 + \|T\mathbf{x}\|_{\mathcal{Y}}^2}$. Define $\Pi_1 : \Gamma(T) \to \mathcal{X}$ and $\Pi_2 : \Gamma(T) \to \mathcal{Y}$ by

$$\Pi_1\big((\mathbf{x}, T\mathbf{x})\big) = \mathbf{x} \qquad \Pi_2\big((\mathbf{x}, T\mathbf{x})\big) = T\mathbf{x}$$

Then Π_1 is continuous and bijective and Π_2 is continuous. Hence $T = \Pi_2\Pi_1^{-1}$ is continuous, by part (e), the inverse mapping theorem.

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Proof of Theorem 1.g: We shall show that $\Gamma(T)$ is closed. Let $\big\{\mathbf{x}_n\big\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Assume that $\big\{\mathbf{x}_n\big\}_{n\in\mathbb{N}}$ converges to $\mathbf{x} \in \mathcal{H}$ and that $\big\{T\mathbf{x}_n\big\}_{n\in\mathbb{N}}$ converges to $\mathbf{y} \in \mathcal{H}$. We must prove that $\mathbf{y} = T\mathbf{x}$. But this follows from the observation that, for all $\mathbf{z} \in \mathcal{H}$,

$$\langle \mathbf{z}, \mathbf{y}\rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle \mathbf{z}, T\mathbf{x}_n\rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle T\mathbf{z}, \mathbf{x}_n\rangle_{\mathcal{H}} = \langle T\mathbf{z}, \mathbf{x}\rangle_{\mathcal{H}} = \langle \mathbf{z}, T\mathbf{x}\rangle_{\mathcal{H}}$$

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