

The Principle of Uniform Boundedness, and Friends

In these notes, unless otherwise stated, \mathcal{X} and \mathcal{Y} are Banach spaces and $T : \mathcal{X} \rightarrow \mathcal{Y}$ is linear and has domain \mathcal{X} .

Theorem 1

(a) T is bounded if and only if

$$T^{-1}\{ \mathbf{y} \in \mathcal{Y} \mid \|\mathbf{y}\|_{\mathcal{Y}} \leq 1 \} = \{ \mathbf{x} \in \mathcal{X} \mid \|T\mathbf{x}\|_{\mathcal{Y}} \leq 1 \}$$

has nonempty interior. (\mathcal{X}, \mathcal{Y} need not be complete.)

(b) **Principle of Uniform Boundedness:** Let $\mathcal{F} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

If, for each $\mathbf{x} \in \mathcal{X}$, $\{ \|T\mathbf{x}\| \mid T \in \mathcal{F} \}$ is bounded,
then $\{ \|T\| \mid T \in \mathcal{F} \}$ is bounded,

(\mathcal{Y} need not be complete.)

(c) If $B : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{C}$ is bilinear and continuous in each variable separately (i.e. $B(\mathbf{x}, \mathbf{y})$ is continuous in \mathbf{x} for each fixed \mathbf{y} and vice versa), then $B(\mathbf{x}, \mathbf{y})$ is jointly continuous (i.e. if $\lim_{n \rightarrow \infty} \mathbf{x}_n = 0$ and $\lim_{n \rightarrow \infty} \mathbf{y}_n = 0$, then $\lim_{n \rightarrow \infty} B(\mathbf{x}_n, \mathbf{y}_n) = 0$).

(d) **Open Mapping Theorem:** If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is surjective (i.e. onto) and if \mathcal{O} is an open subset of \mathcal{X} , then $T\mathcal{O} = \{ T\mathbf{x} \mid \mathbf{x} \in \mathcal{O} \}$ is an open subset of \mathcal{Y} .

(e) **Inverse Mapping Theorem:** If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is bijective (i.e. 1-1 and onto), then T^{-1} is bounded.

(f) **Closed Graph Theorem:** The graph of T is defined to be

$$\Gamma(T) = \{ (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y} \mid \mathbf{y} = T\mathbf{x} \}$$

Then

$$T \text{ is bounded} \iff \Gamma(T) \text{ is closed}$$

(g) **Hellinger–Toeplitz Theorem:** Let T be an everywhere defined linear operator on \mathcal{H} that obeys $\langle \mathbf{x}, T\mathbf{y} \rangle = \langle T\mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Then T is bounded.

Remark 2 The linearity of T is crucial for Theorem 1. For part (c), for example, the function $T : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ defined by

$$T(x, y) = \begin{cases} \frac{|xy|}{|x|^2+|y|^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous in x and y separately, but not in (x, y) jointly. (The limit as $(x, y) \rightarrow (0, 0)$ with $x = y$ is $\frac{1}{2}$, while the limit as $(x, y) \rightarrow (0, 0)$ with $x = 0$ is 0.)

Proof of Theorem 1.a:

\Leftarrow : By hypothesis, there are $\varepsilon > 0$ and $\mathbf{x}_0 \in \mathcal{X}$ such that

$$\|T(\mathbf{x} + \mathbf{x}_0)\|_{\mathcal{Y}} \leq 1 \quad \text{for all } \mathbf{x} \in \mathcal{X} \text{ with } \|\mathbf{x}\|_{\mathcal{X}} < \varepsilon$$

Since T is linear, if $\mathbf{x} \in \mathcal{X}$ with $\|\mathbf{x}\|_{\mathcal{X}} < \varepsilon$, then

$$\|T\mathbf{x}\|_{\mathcal{Y}} = \|T(\mathbf{x} + \mathbf{x}_0) - T\mathbf{x}_0\|_{\mathcal{Y}} \leq \|T(\mathbf{x} + \mathbf{x}_0)\|_{\mathcal{Y}} + \|T\mathbf{x}_0\|_{\mathcal{Y}} \leq 1 + \|T\mathbf{x}_0\|_{\mathcal{Y}}$$

So, for any $0 \neq \mathbf{x} \in \mathcal{X}$ and any $0 < \varepsilon' < \varepsilon$,

$$\|T\mathbf{x}\|_{\mathcal{Y}} = \frac{\|\mathbf{x}\|_{\mathcal{X}}}{\varepsilon'} \left\| T\left(\varepsilon' \frac{\mathbf{x}}{\|\mathbf{x}\|_{\mathcal{X}}}\right) \right\|_{\mathcal{Y}} \leq \frac{1}{\varepsilon'} [1 + \|T\mathbf{x}_0\|_{\mathcal{Y}}] \|\mathbf{x}\|_{\mathcal{X}}$$

Hence $\|T\| \leq \frac{1}{\varepsilon} [1 + \|T\mathbf{x}_0\|_{\mathcal{Y}}]$.

\Rightarrow : By hypothesis $\|T\|$ is finite. Hence

$$\mathbf{x} \in \mathcal{X}, \|\mathbf{x}\|_{\mathcal{X}} < \frac{1}{\|T\|} \implies \|T\mathbf{x}\|_{\mathcal{Y}} \leq \|T\| \|\mathbf{x}\|_{\mathcal{X}} < 1$$

So $\{ \mathbf{x} \in \mathcal{X} \mid \|\mathbf{x}\|_{\mathcal{X}} < \frac{1}{\|T\|} \} \subset \{ \mathbf{x} \in \mathcal{X} \mid \|T\mathbf{x}\|_{\mathcal{Y}} \leq 1 \}$ and the latter has nonempty interior. ■

Definition 3 (Nowhere Dense) A subset S of a metric space \mathcal{M} is said to be *nowhere dense* if its closure \bar{S} has empty interior. Equivalently, S is nowhere dense if it is not dense in any open ball of \mathcal{M} .

Theorem 4 (Baire Category Theorem) *Let \mathcal{M} be a nonempty complete metric space. Then \mathcal{M} cannot be written as the union of a countable number of nowhere dense subsets.*

Proof: The proof is by contradiction. Suppose that $\mathcal{M} = \bigcup_{n=1}^{\infty} S_n$, with each S_n nowhere dense. We shall construct a Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ whose limit $x \notin \bigcup_{n=1}^{\infty} S_n$.

- Since S_1 is nowhere dense, $\overline{S_1} \neq \mathcal{M}$. So there must exist some $x_1 \in \mathcal{M}$ and some $0 < r_1 < \frac{1}{2}$ such that $B_{r_1}(x_1) \cap S_1 = \emptyset$. Here, as usual, $B_{r_1}(x_1)$ denotes the open ball of radius r_1 centred on x_1 .
- Since S_2 is nowhere dense, $\overline{S_2}$ may not contain all of $B_{r_1}(x_1)$. So there must exist some $x_2 \in B_{r_1}(x_1)$ and some $0 < r_2 < \frac{1}{4}$ and $B_{r_2}(x_2) \cap S_2 = \emptyset$. Possibly shrinking r_2 , we may assume that $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1)$.
- Continue in this way to construct inductively, for each $n \in \mathbb{N}$, an $x_{n+1} \in B_{r_n}(x_n)$ and a $0 < r_{n+1} < \frac{1}{2^{n+1}}$ such that $B_{r_{n+1}}(x_{n+1}) \cap S_{n+1} = \emptyset$ and $\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n)$.

Write, for each $n \in \mathbb{N}$, $B_n = B_{r_n}(x_n)$. We have now constructed

$$\begin{array}{c}
 x_1 \in B_1 \quad \text{with } B_1 \cap S_1 = \emptyset \\
 \cup \\
 \overline{B_2} \\
 \cup \\
 x_2 \in B_2 \quad \text{with } B_2 \cap S_2 = \emptyset \\
 \cup \\
 \overline{B_3} \\
 \cup \\
 x_3 \in B_3 \quad \text{with } B_3 \cap S_3 = \emptyset \\
 \vdots
 \end{array}$$

For each $n \in \mathbb{N}$, $x_{n+1} \in B_{r_n}(x_n)$. In particular, $d(x_n, x_{n+1}) < r_n \leq \frac{1}{2^n}$. Consequently $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy. As \mathcal{M} is complete, it has a limit, x . Since $x_n \in B_m$ for all $n \geq m$, we have $x \in \overline{B_m}$ for all $m \geq 2$ and hence $x \in B_\ell$ for all $\ell \in \mathbb{N}$. So $x \notin S_\ell$ for all $\ell \in \mathbb{N}$. ■

Proof of Theorem 1.b: Set $\mathcal{M} = \mathcal{X}$ and, for each $n \in \mathbb{N}$,

$$S_n = \left\{ \mathbf{x} \in \mathcal{X} \mid \|T\mathbf{x}\|_{\mathcal{Y}} \leq n \text{ for all } T \in \mathcal{F} \right\} = \bigcap_{T \in \mathcal{F}} T^{-1} \left\{ \mathbf{y} \in \mathcal{Y} \mid \|\mathbf{y}\|_{\mathcal{Y}} \leq n \right\}$$

By hypothesis $\mathcal{M} = \bigcup_{n=1}^{\infty} S_n$. Since each $T \in \mathcal{F}$ is continuous, each S_n , $n \in \mathbb{N}$ is closed. By the Baire category theorem, there is an $n_0 \in \mathbb{N}$ such that S_{n_0} has nonempty interior. Just as in the proof of of Theorem 1.a, there is a $\varepsilon > 0$ and an $\mathbf{x}_0 \in \mathcal{X}$ such that

$$\|T\| \leq \frac{1}{\varepsilon} [n_0 + \|T\mathbf{x}_0\|_{\mathcal{Y}}] \leq \frac{1}{\varepsilon} \left[n_0 + \sup_{T' \in \mathcal{F}} \|T'\mathbf{x}_0\|_{\mathcal{Y}} \right] < \infty$$

for all $T \in \mathcal{F}$. ■

Proof of Theorem 1.c: Let $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{0}$ and $\lim_{n \rightarrow \infty} \mathbf{y}_n = \mathbf{0}$. For each $n \in \mathbb{N}$, define $T_n : \mathcal{Y} \rightarrow \mathbb{C}$ by $T_n \mathbf{y} = B(\mathbf{x}_n, \mathbf{y})$. Since T_n is continuous (recall that, for each fixed $\mathbf{x} \in \mathcal{X}$, $B(\mathbf{x}, \mathbf{y})$ is continuous in \mathbf{y}) and linear, it is bounded. For each fixed $\mathbf{y} \in \mathcal{Y}$, $\lim_{n \rightarrow \infty} T_n \mathbf{y} = 0$, since $B(\mathbf{x}, \mathbf{y})$ is continuous in \mathbf{x} when \mathbf{y} is held fixed. Thus, for each fixed $\mathbf{y} \in \mathcal{Y}$, $\{ T_n \mathbf{y} \mid n \in \mathbb{N} \}$ is bounded. Hence, by the Principle of Uniform Boundedness, $\tau = \sup \{ \|T_n\| \mid n \in \mathbb{N} \} < \infty$ so that,

$$|B(\mathbf{x}_n, \mathbf{y}_n)| = \|T_n \mathbf{y}_n\| \leq \|T_n\| \|\mathbf{y}_n\| \leq \tau \|\mathbf{y}_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

■

Proof of Theorem 1.d:

Step 1: I claim that it suffices to prove that there is an $\varepsilon > 0$ such that

$$B_\varepsilon(\mathbf{0}) \subset TB_2(\mathbf{0}) = \{ T\mathbf{x} \mid \|\mathbf{x}\|_{\mathcal{X}} < 2 \} \tag{1}$$

Because then,

- for every $r > 0$, $B_{\frac{1}{2}r\varepsilon}(\mathbf{0}) \subset TB_r(\mathbf{0})$, by scaling, so that $\mathbf{0}$ is in the interior of $TB_r(\mathbf{0})$
- and for every $\mathbf{x} \in \mathcal{X}$ and every $r > 0$, $B_{\frac{1}{2}r\varepsilon}(T\mathbf{x}) \subset TB_r(\mathbf{x})$, by translating, so that $T\mathbf{x}$ is in the interior of $TB_r(\mathbf{x})$
- so that, if we choose, for each $\mathbf{x} \in \mathcal{O}$, $r_{\mathbf{x}} > 0$ sufficiently small that $B_{r_{\mathbf{x}}}(\mathbf{x}) \subset \mathcal{O}$,

$$T\mathcal{O} = \bigcup_{\mathbf{x} \in \mathcal{O}} \{T\mathbf{x}\} \subset \bigcup_{\mathbf{x} \in \mathcal{O}} (TB_{r_{\mathbf{x}}}(\mathbf{x}))^\circ \subset \bigcup_{\mathbf{x} \in \mathcal{O}} (T\mathcal{O})^\circ = (T\mathcal{O})^\circ$$

Step 2: We now prove that there is an $\varepsilon > 0$ obeying (1).

$$\begin{aligned} T \text{ surjective} &\implies \mathcal{Y} = \bigcup_{n \in \mathbb{N}} TB_n(\mathbf{0}) \\ &\implies \overline{TB_n(\mathbf{0})} \text{ has a nonempty interior for some } n \in \mathbb{N} \\ &\quad \text{(by the Baire Category Theorem)} \\ &\implies B_\varepsilon(\mathbf{0}) \subset \overline{TB_1(\mathbf{0})} \text{ for some } \varepsilon > 0 \\ &\quad \text{(by scaling and translating)} \end{aligned}$$

So it suffices to prove that $\overline{TB_1(\mathbf{0})} \subset TB_2(\mathbf{0})$. So let $\mathbf{y} \in \overline{TB_1(\mathbf{0})}$. Then

- $\exists \mathbf{x}_1 \in B_1(\mathbf{0})$ such that $\mathbf{y} - T\mathbf{x}_1 \in B_{\frac{\varepsilon}{2}}(\mathbf{0}) \subset \overline{TB_{\frac{1}{2}}(\mathbf{0})}$ and
- $\exists \mathbf{x}_2 \in B_{\frac{1}{2}}(\mathbf{0})$ such that $\mathbf{y} - T\mathbf{x}_1 - T\mathbf{x}_2 \in B_{\frac{\varepsilon}{4}}(\mathbf{0}) \subset \overline{TB_{\frac{1}{4}}(\mathbf{0})}$
- and so on

Set $\mathbf{x} = \sum_{n=1}^{\infty} \mathbf{x}_n$. Then $\|\mathbf{x}\|_{\mathcal{X}} \leq \sum_{n=1}^{\infty} \|\mathbf{x}_n\|_{\mathcal{X}} < \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2$ so that $\mathbf{x} \in B_2(\mathbf{0})$ and $\mathbf{y} = \sum_{n=1}^{\infty} T\mathbf{x}_n = T\mathbf{x} \in TB_2(\mathbf{0})$. ■

Proof of Theorem 1.e: Since T is linear and bijective, it has a linear inverse, T^{-1} , defined on all of \mathcal{Y} . If \mathcal{O} is any open subset of \mathcal{X} , then

$$(T^{-1})^{-1}(\mathcal{O}) = T(\mathcal{O})$$

is open, by part (d), the open mapping theorem. So T^{-1} is continuous, by definition. ■

Proof of Theorem 1.f:

\Rightarrow : By hypothesis T is bounded. Suppose that the sequence $\{(\mathbf{x}_n, \mathbf{y}_n)\}_{n \in \mathbb{N}} \subset \Gamma(T)$ converges to $(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$. That is, assume that $\mathbf{x} = \lim_{n \rightarrow \infty} \mathbf{x}_n$ and $\mathbf{y} = \lim_{n \rightarrow \infty} \mathbf{y}_n$. Then

$$\mathbf{y} = \lim_{n \rightarrow \infty} \mathbf{y}_n = \lim_{n \rightarrow \infty} T\mathbf{x}_n = T \lim_{n \rightarrow \infty} \mathbf{x}_n = T\mathbf{x}$$

Thus $(\mathbf{x}, \mathbf{y}) \in \Gamma(T)$ and $\Gamma(T)$ is closed.

\Leftarrow : By hypothesis $\Gamma(T)$ is closed. As a result, $\Gamma(T)$ is a Banach space with the norm $\|(\mathbf{x}, T\mathbf{x})\|_{\Gamma(T)} = \sqrt{\|\mathbf{x}\|_{\mathcal{X}}^2 + \|T\mathbf{x}\|_{\mathcal{Y}}^2}$. Define $\Pi_1 : \Gamma(T) \rightarrow \mathcal{X}$ and $\Pi_2 : \Gamma(T) \rightarrow \mathcal{Y}$ by

$$\Pi_1((\mathbf{x}, T\mathbf{x})) = \mathbf{x} \quad \Pi_2((\mathbf{x}, T\mathbf{x})) = T\mathbf{x}$$

Then Π_1 is continuous and bijective and Π_2 is continuous. Hence $T = \Pi_2\Pi_1^{-1}$ is continuous, by part (e), the inverse mapping theorem. ■

Proof of Theorem 1.g: We shall show that $\Gamma(T)$ is closed. Let $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{H} . Assume that $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ converges to $\mathbf{x} \in \mathcal{H}$ and that $\{T\mathbf{x}_n\}_{n \in \mathbb{N}}$ converges to $\mathbf{y} \in \mathcal{H}$. We must prove that $\mathbf{y} = T\mathbf{x}$. But this follows from the observation that, for all $\mathbf{z} \in \mathcal{H}$,

$$\langle \mathbf{z}, \mathbf{y} \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \langle \mathbf{z}, T\mathbf{x}_n \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \langle T\mathbf{z}, \mathbf{x}_n \rangle_{\mathcal{H}} = \langle T\mathbf{z}, \mathbf{x} \rangle_{\mathcal{H}} = \langle \mathbf{z}, T\mathbf{x} \rangle_{\mathcal{H}}$$

■