Review of Unbounded Operators

Definition 1  Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $T : \mathcal{D}(T) \subset \mathcal{H}_1 \to \mathcal{H}_2$ be a linear operator with domain $\mathcal{D}(T)$.

(a) The graph of $T$ is

$$\Gamma(T) = \{ (\varphi, T\varphi) \mid \varphi \in \mathcal{D}(T) \} \subset \mathcal{H}_1 \times \mathcal{H}_2$$

(b) The operator $T$ is said to be **closed** if $\Gamma(T)$ is a closed subset of $\mathcal{H}_1 \times \mathcal{H}_2$. That is, if

$$\lim_{n \to \infty} \varphi_n = \varphi, \quad \lim_{n \to \infty} T\varphi_n = \psi \quad \implies \quad (\varphi, \psi) \in \Gamma(T)$$

(c) The operator $T$ is said to be **closable** if $\overline{\Gamma(T)}$ is the graph of an operator. That is,

\begin{align*}
(\varphi_1, \psi_1), \ (\varphi_2, \psi_2) &\in \overline{\Gamma(T)} \implies \psi_1 = \psi_2 \\
\text{i.e. } \lim_{n \to \infty} \varphi_n = \lim_{n \to \infty} \varphi'_n, \ \lim_{n \to \infty} T\varphi_n, \ \lim_{n \to \infty} T\varphi'_n \ 	ext{exist} \implies \lim_{n \to \infty} T\varphi_n = \lim_{n \to \infty} T\varphi'_n \\
\text{i.e. } \lim_{n \to \infty} \varphi_n = 0, \ \lim_{n \to \infty} T\varphi_n \ 	ext{exist} \implies \lim_{n \to \infty} T\varphi_n = 0
\end{align*}

If $T$ is closable, the operator $\overline{T}$ defined by $\Gamma(\overline{T}) = \overline{\Gamma(T)}$ is called the **closure** of $T$. The domain of $\overline{T}$ is

$$\mathcal{D}(\overline{T}) = \{ \varphi \in \mathcal{H}_1 \mid \exists \{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(T) \text{ s.t. } \varphi = \lim_{n \to \infty} \varphi_n, \ \lim_{n \to \infty} T\varphi_n \ 	ext{exists} \}$$

If $\varphi = \lim_{n \to \infty} \varphi_n$ and $\lim_{n \to \infty} T\varphi_n$ exists, then $\overline{T}\varphi = \lim_{n \to \infty} T\varphi_n$.

(d) A linear operator $S$ is said to be an **extension** of $T$ if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $S\varphi = T\varphi$ for all $\varphi \in \mathcal{D}(T)$. We then write $T \subset S$.

Lemma 2  If $A : \varphi(x) \in \mathcal{D}(A) \subset L^2(M, \mu) \mapsto a(x)\varphi(x) \in L^2(M, \mu)$ is a multiplication operator, then $A$ is closable and its closure is again multiplication by $a(x)$.

Example 3  It is easy to construct, using an algebraic basis, a linear operator whose domain is the entire Hilbert space, but which is unbounded. (We are of course assuming that the Hilbert space is infinite dimensional.) By the closed graph theorem, this operator cannot be closed. So it provides an extreme example of an operator which is not closable.
Example 4  On the other hand, it is also possible for an operator to have many closed extensions. Here is an example. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$ and the operator is

$$\mathcal{D}(A) = \{ f \in C_0^\infty(\mathbb{R}) \mid \int_{-\infty}^{\infty} f(k) \, dk = \int_{-\infty}^{\infty} kf(k) \, dk = 0 \} \quad (Af)(k) = (1 + k^2)f(k)$$

If you take Fourier transforms, this operator becomes the differential operator $-\frac{d^2}{dx^2} + 1$ with “initial conditions” $\hat{f}(0) = \frac{df}{dx}(0) = 0$. Set

$$p_0(k) = \frac{1}{1+k^2} \quad p_1(k) = \frac{k}{1+k^2}$$

The the closure of $A$ is

$$\mathcal{D}(\bar{A}) = \{ f \in L^2(\mathbb{R}) \mid (1 + k^2)f \in L^2(\mathbb{R}), (1 + k^2)f \perp p_0, p_1 \} \quad (\bar{A}f)(k) = (1 + k^2)f(k)$$

Choose any nonzero $p, q \in \text{span}\{p_0, p_1\}$ and a nonzero $p^\perp \in \text{span}\{p_0, p_1\}$ which is perpendicular to $p$. The following are all closed extensions of $A$.

$$\mathcal{D}(A_1) = \{ f \in L^2(\mathbb{R}) \mid (1 + k^2)f \in L^2(\mathbb{R}), (1 + k^2)f \perp p \} \quad (A_1f)(k) = (1 + k^2)f(k)$$

$$\mathcal{D}(A_2) = \{ f \in L^2(\mathbb{R}) \mid (1 + k^2)f \in L^2(\mathbb{R}) \} \quad (A_2f)(k) = (1 + k^2)f(k)$$

$$\mathcal{D}(A_3) = \mathcal{D}(A_1) = \{ \alpha \frac{p^\perp}{1+k^2} + f \mid \alpha \in \mathbb{C}, (1 + k^2)f \in \{p_0, p_1\}^\perp \}$$

$$A_3(\alpha \frac{p^\perp}{1+k^2} + f) = \alpha q + (1 + k^2)f$$

Definition 5  Let $T$ be a densely defined linear operator on a Hilbert space $\mathcal{H}$. The adjoint $T^*$ of $T$ is defined by

$$\varphi \in \mathcal{D}(T^*) \iff \exists \eta \in \mathcal{H} \text{ s.t. } \langle T\psi, \varphi \rangle_{\mathcal{H}} = \langle \psi, \eta \rangle_{\mathcal{H}} \quad \forall \psi \in \mathcal{D}(T)$$

$$T^*\varphi = \eta \iff \langle T\psi, \varphi \rangle_{\mathcal{H}} = \langle \psi, \eta \rangle_{\mathcal{H}} \quad \forall \psi \in \mathcal{D}(T)$$

Remark 6

(a) The $\eta$ of the last definition is unique when it exists, because $\mathcal{D}(T)$ is dense.

(b) By the Riesz representation theorem, $\varphi \in \mathcal{D}(T^*)$ if and only if there is a constant $K_\varphi$ such that $|\langle T\psi, \varphi \rangle_{\mathcal{H}}| \leq K_\varphi \|\psi\|_{\mathcal{H}}$ for all $\psi \in \mathcal{D}(T)$.

(c) $0 \in \mathcal{D}(T^*)$ always.

(d) If $S \subset T$, then $T^* \subset S^*$. 

November 20, 2018  Review of Unbounded Operators
The following example shows that it is possible to have $\mathcal{D}(T^*) = \{0\}$. Let
- $\mathcal{H} = L^2(\mathbb{R})$,
- $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and
- for each $k \in \mathbb{N}$, $f_k(x) = e^{ikx}$. Note that $f_k \notin \mathcal{H}$.

We define the domain of $T$ to be $\mathcal{D}(T) = \mathcal{B}_0(\mathbb{R})$, the set of all bounded Borel functions on $\mathbb{R}$ that are of compact support. This domain is dense in $L^2(\mathbb{R})$. For $\varphi \in \mathcal{D}(T)$,

$$T\varphi = \sum_{n=1}^{\infty} \left[ \int_{-\infty}^{\infty} f_n(x)\varphi(x) \, dx \right] e_n$$

**Step 1:** We first check that $T$ is well-defined. Let $\varphi \in \mathcal{D}(T)$. Then there is some integer $m$ such that $\varphi(x)$ vanishes outside of $[-m\pi, m\pi]$. Then, for each $k \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} f_k(x)\varphi(x) \, dx = \int_{-m\pi}^{m\pi} f_k(x)\varphi(x) \, dx = \int_{-\pi}^{\pi} e^{ikmt} m\varphi(mt) \, dt$$

is the $km$th Fourier coefficient of the function $m\varphi(mt)$. Since the sum of the squares of all of the Fourier coefficients is, up to a factor of $2\pi$, the $L^2$ norm of $m\varphi(mt)$, which is finite, $T$ is well-defined.

**Step 2:** We now check that $\mathcal{D}(T^*) = \{0\}$. Let $\psi \in \mathcal{D}(T^*)$ and $\varphi \in \mathcal{D}(T) = \mathcal{B}_0(\mathbb{R})$. Pick an $m \in \mathbb{N}$ with $\varphi(x)$ vanishing except for $x$ in $[-m\pi, m\pi]$. Then

$$\langle \varphi, T^*\psi \rangle = \langle T\varphi, \psi \rangle = \sum_{n=1}^{\infty} \left[ \int_{-\infty}^{\infty} f_n(x)\varphi(x) \, dx \right] \langle e_n, \psi \rangle = \sum_{n=1}^{\infty} \langle \varphi, (e_n, \psi)^T \rangle$$

Since

$$\int_{-m\pi}^{m\pi} f_k(x)f_\ell(x) \, dx = \int_{-m\pi}^{m\pi} e^{i(\ell-k)x} \, dx = \begin{cases} 2m\pi & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

The series $\sum_{n=1}^{\infty} \langle e_n, \psi \rangle \overline{T_n}$ converges in $\mathcal{H}$ and

$$\langle \varphi, T^*\psi \rangle = \left\langle \varphi, \sum_{n=1}^{\infty} (e_n, \psi)^T \overline{T_n} \right\rangle$$

This is true for all bounded Borel functions $\varphi$ supported in $[-m\pi, m\pi]$ so that

$$T^*\psi = \sum_{n=1}^{\infty} (e_n, \psi)^T \overline{T_n} \quad \text{a.e. on } [-m\pi, m\pi]$$

and

$$\infty > \|T^*\psi\|_{L^2(\mathbb{R})} \geq \|T^*\psi\|_{L^2([-m\pi, m\pi])} = \sum_{n=1}^{\infty} |\langle e_n, \psi \rangle|^2 (2m\pi)$$

Since this is true for all $m$, we must have $\langle e_n, \psi \rangle = 0$ for all $n \in \mathbb{N}$ and hence $\psi = 0$. 

November 20, 2018  Review of Unbounded Operators 3
**Theorem 8** Let $\mathcal{H}$ be a Hilbert space and let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator. Then

(a) $T^*$ is closed.
(b) $T$ is closable if and only if $D(T^*)$ is dense. If so, $\bar{T} = T^{**}$.
(c) If $T$ is closable, then $(\bar{T})^* = T^*$.

**Definition 9** Let $\mathcal{H}$ be a Hilbert space and let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator.

(a) $T$ is called **symmetric** if

$$\langle \varphi, T\psi \rangle = \langle T\psi, \varphi \rangle \quad \text{for all } \varphi, \psi \in D(T)$$

That is, if $T \subset T^*$.
(b) $T$ is called **self-adjoint** if $T = T^*$.

**Remark 10**

(a) If $T$ is self-adjoint, then it is automatically closed.

(b) Let $(M, \mu)$ be a measure space. If $f(m)$ is a real-valued, measurable function on $M$, then the operator of multiplication by $f$, with domain $\{ \psi \in L^2(M, \mu) \mid f\psi \in L^2(M, \mu) \}$ is self-adjoint.

**Theorem 11** Let $\mathcal{H}$ be a Hilbert space. A densely defined, symmetric operator $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint if and only if the ranges of $\text{ran}(T - i1l) = \text{ran}(T + i1l) = \mathcal{H}$.

**Definition 12** Let $T$ be a densely defined, closed operator on a Hilbert space $\mathcal{H}$.

(a) The **resolvent set**, $\rho(T)$, of $T$ is the set of all complex numbers $\lambda$ such that $\lambda 1l - T$ is a bijection (between $D(T)$ and $\mathcal{H}$) with bounded inverse.

(b) The **resolvent** of $T$ at $\lambda \in \rho(T) \subset \mathbb{C}$ is $R_\lambda(T) = (\lambda 1l - T)^{-1}$.

(c) The **spectrum** of $T$ is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

(d) The complex number $\lambda$ is in the **point spectrum**, $\sigma_p(T)$, of $T$ if $\lambda 1l - T$ is not injective. That is, if there is a nonzero vector $x \in \mathcal{H}$ such that $Tx = \lambda x$. Then $x$ is said to be an eigenvector of $T$ with eigenvalue $\lambda$.

(e) The complex number $\lambda$ is in the **residual spectrum**, $\sigma_r(T)$, of $T$ if $\lambda 1l - T$ is injective but the range of $T$ is not dense in $\mathcal{B}$. 

November 20, 2018 Review of Unbounded Operators 4
Example 13  Here is an example which shows, firstly, that an unbounded operator $T$ may have $\sigma(T) = \emptyset$ and secondly that “just changing the domain of an operator” can change its spectrum. The Hilbert space $\mathcal{H} = L^2([0,1])$. Denote by $AC[0,1]$ the set of all functions $f : [0,1] \to \mathbb{C}$ of the form $f(x) = C + \int_0^x \varphi(t) \, dt$ with $C \in \mathbb{C}$ and $\varphi \in L^2([0,1])$. For such a function, define $\frac{df}{dx}(x) = \varphi(x)$. All of the operators in this example will be closed operators with domain contained in $AC[0,1]$ and all operators will have the action $f(x) \mapsto i\frac{df}{dx}(x)$.

Here are the examples.
(a) If $\mathcal{D}(T) = AC[0,1]$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \mathbb{C}$. In fact $\sigma_p(T) = \mathbb{C}$, since, for any $\lambda \in \mathbb{C}$, $e^{-i\lambda x}$ is an eigenfunction for $T$ with eigenvalue $\lambda$.
(b) If $\mathcal{D}(T) = \{ f \in AC[0,1] \mid f(0) = 0 \}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \emptyset$. In fact, for any $\lambda \in \mathbb{C}$, the resolvent operator $(\lambda \mathbb{1} - T)^{-1}$ is

$$
(R_\lambda(T)\psi)(x) = i \int_0^x e^{-i\lambda(x-t)}\psi(t) \, dt
$$

(c) Let $\alpha \in \mathbb{C}$ be nonzero. If $\mathcal{D}(T) = \{ f \in AC[0,1] \mid f(0) = \alpha f(1) \}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \{ -i\ln\alpha + 2k\pi \mid k \in \mathbb{Z} \}$. Again the spectrum consists solely of eigenvalues. If $\lambda = -i\ln\alpha + 2k\pi$ for some $k \in \mathbb{Z}$, then $e^{-i\lambda x}$ is an eigenfunction for $T$ with eigenvalue $\lambda$. For $\lambda$ not of the form $-i\ln\alpha + 2k\pi$ for all $k \in \mathbb{Z}$, the resolvent operator $(\lambda \mathbb{1} - T)^{-1}$ is

$$
(R_\lambda(T)\psi)(x) = \int_0^1 G_\lambda(x,t)\psi(t) \, dt
$$

with

$$
G_\lambda(x,t) = \begin{cases} 
\frac{ie^{i\lambda(t-x)}}{1-\alpha e^{-i\lambda}} & \text{if } x < t \\
\frac{ie^{i\lambda(t-x)}}{1-\alpha e^{-i\lambda}} & \text{if } x > t
\end{cases}
$$

Theorem 14  (Cayley Transform) Let $\mathcal{H}$ be a Hilbert space.

(a) If $A = A^*$ is a self-adjoint operator on $\mathcal{H}$, then $C_+(A) = (A - i\mathbb{1})(A + i\mathbb{1})^{-1}$ exists and is a unitary operator on $\mathcal{H}$. Furthermore, $1 \notin \sigma_p(C_+(A))$.

(b) If $U$ is a unitary operator on $\mathcal{H}$ and $1 \notin \sigma_p(U)$, then $C_-(U) = i(\mathbb{1} + U)(\mathbb{1} - U)^{-1}$ exists and is self-adjoint.

(c) $C_+$ and $C_-$ are inverses. That is, $C_+(C_-(U)) = U$ and $C_-(C_+(A)) = A$ for all unitary $U$ with $1 \notin \sigma_p(U)$ and all self-adjoint $A$.

(d) If $T$ is a symmetric operator with domain $D(T)$, then $C_+(T) = (T - i\mathbb{1})(T + i\mathbb{1})^{-1}$ is a well-defined isometric operator with domain $R(T + i\mathbb{1})$ and range $R(T - i\mathbb{1})$. Furthermore, $1 \notin \sigma_p(C_+(T))$ and $T = C_-(C_+(T))$. 

November 20, 2018  Review of Unbounded Operators 5
Remark 15 Let $T$ be a symmetric operator. By the last theorem, there is a one-to-one correspondence between self-adjoint extensions of $T$ and unitary operators $U$ that
(a) extend the isometric operator $C_+(T)$ and
(b) obey $1 \notin \sigma_p(U)$.
First forget about the condition that $1 \notin \sigma_p(U)$. It’s a simple matter to construct all unitary operators that extend $C_+(T)$. Since the closure $\overline{C_+(T)}$ is an isometric operator with domain $\overline{R(T + i\mathbb{I})}$ and range $\overline{R(T - i\mathbb{I})}$, there is a one-to-one correspondence between such unitary extensions and unitary maps $\tilde{U} : R(T + i\mathbb{I})^\perp \rightarrow R(T - i\mathbb{I})^\perp$. Now back to condition (b). The following lemma shows it is automatically satisfied!

Lemma 16 Let $\mathcal{H}$ be a Hilbert space and $T : D(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a densely defined, symmetric operator. Let $U$ be a unitary extension of $C_+(T)$. Then $1 \notin \sigma_p(U)$.

Definition 17 An operator is called essentially self-adjoint if its closure is self-adjoint. A subset $D \subset \mathcal{H}$ is called a core for the self-adjoint operator $T$ if $\overline{T|D} = T$.

Theorem 18 Let $T$ be a densely defined operator on a Hilbert space $\mathcal{H}$. Then

\[ T \text{ is essentially self-adjoint} \iff T \text{ has a unique self-adjoint extension} \]

Theorem 19 Let $T$ be a symmetric operator on a Hilbert space $\mathcal{H}$.

(a) If $\dim (T + i) = \dim (T - i) = 0$, then $T$ has a unique self-adjoint extension, and so is essentially self-adjoint.

(b) If $\dim (T + i) = \dim (T - i) \geq 1$, then $T$ has infinitely many distinct self-adjoint extensions.

(c) If $\dim (T + i) \neq \dim (T - i)$, then $T$ has no self-adjoint extensions.

Theorem 20 (Spectral Theorem - Multiplication Operator Version)

Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. There exist
\begin{itemize}
  \item a measure space $(\mathcal{M}, \Sigma, \mu)$,
  \item a measurable function $a : \mathcal{M} \rightarrow \mathbb{R}$, and
  \item a unitary operator $U : \mathcal{H} \rightarrow L^2(\mathcal{M}, \Sigma, \mu)$
\end{itemize}
such that $UAU^{-1}$ is the operator of multiplication by the function $a(m)$, with domain \( \{ \varphi \in L^2(M, \mu) \mid a\varphi \in L^2(M, \Sigma, \mu) \} \). If $\mathcal{H}$ is separable, $\mu$ can be chosen to be a finite measure.
Definition 21 (Projection Valued Measure)

(a) Denote by $\mathcal{B}_{\mathbb{R}}$ the $\sigma$–algebra of Borel subsets of $\mathbb{R}$ and by $\mathcal{L}(\mathcal{H})$ the set of a bounded operators on $\mathcal{H}$. A projection valued measure is a map $E : \mathcal{B}_{\mathbb{R}} \to \mathcal{L}(\mathcal{H})$ that obeys the following conditions.

(i) For each $B \in \mathcal{B}_{\mathbb{R}}$, the operator $E(B)$ is an orthogonal projection on some closed subspace of $\mathcal{H}$. That is, $E(B)^2 = E(B)$ and $E(B) = E(B)^*$. 
(ii) $E(\emptyset) = 0$ and $E(\mathbb{R}) = 1$
(iii) If $\{B_n\}_{n \in \mathbb{N}}$ is a countable family of disjoint Borel subsets of $\mathbb{R}$, then

$$E\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{N \to \infty} \sum_{n=1}^{N} E(B_n)$$
(iv) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$. (This condition is a consequence of (i) and (iii).)

(b) Given a projection valued measure $E$ and a Borel function $g$ (not necessarily bounded), the operator $\int g(\lambda) \, dE(\lambda)$ is defined by

$$D\left(\int g(\lambda) \, dE(\lambda)\right) = \left\{ \psi \in \mathcal{H} \left| \int |g(\lambda)|^2 \, d \langle \psi, E(\lambda) \psi \rangle < \infty \right. \right\}$$
$$\left\langle \psi, \int g(\lambda) \, dE(\lambda) \, \varphi \right\rangle = \int g(\lambda) \, d \langle \psi, E(\lambda) \varphi \rangle$$

Theorem 22 (Spectral Theorem - Projection–valued Measure Version)

There is a 1–1 correspondence between self–adjoint operators and projection–valued measures $A \leftrightarrow E_A$ such that

$$A = \int \lambda \, dE_A(\lambda)$$