Review of Unbounded Operators

Definition 1 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $T : \mathcal{D}(T) \subset \mathcal{H}_1 \to \mathcal{H}_2$ be a linear operator with domain $\mathcal{D}(T)$.

(a) The graph of T is

$$\Gamma(T) = \left\{ \ (\varphi\,, T\varphi) \ \middle| \ \varphi \in \mathcal{D}(T) \ \right\} \subset \mathcal{H}_1 \times \mathcal{H}_2$$

(b) The operator T is said to be *closed* if $\Gamma(T)$ is a closed subset of $\mathcal{H}_1 \times \mathcal{H}_2$. That is, if

$$\lim_{n\to\infty} \varphi_n = \varphi, \ \lim_{n\to\infty} T\varphi_n = \psi \qquad \Longrightarrow \qquad (\varphi, \psi) \in \Gamma(T)$$

(c) The operator T is said to be *closable* if $\overline{\Gamma(T)}$ is the graph of an operator. That is,

$$\begin{aligned} & (\varphi, \psi_1),\ (\varphi, \psi_2) \in \overline{\Gamma(T)} \implies \psi_1 = \psi_2 \\ \text{i.e.} \quad & \lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \varphi'_n,\ \lim_{n\to\infty} T\varphi_n,\ \lim_{n\to\infty} T\varphi'_n \text{ exist} \implies \lim_{n\to\infty} T\varphi_n = \lim_{n\to\infty} T\varphi'_n \\ \text{i.e.} \quad & \lim_{n\to\infty} \varphi_n = \mathbf{0},\ \lim_{n\to\infty} T\varphi_n \text{ exists} \implies \lim_{n\to\infty} T\varphi_n = \mathbf{0} \end{aligned}$$

If T is closable, the operator $\bar{T}$ defined by $\Gamma(\bar{T}) = \overline{\Gamma(T)}$ is called the *closure* of T. The domain of $\bar{T}$ is

$$\mathcal{D}(\bar{T}) = \left\{ \ \varphi \in \mathcal{H}_1 \ \middle| \ \exists \ \{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(T) \text{ s.t. } \varphi = \lim_{n\to\infty} \varphi_n,\ \lim_{n\to\infty} T\varphi_n \text{ exists } \right\}$$

If $\varphi = \lim_{n\to\infty} \varphi_n$ and $\lim_{n\to\infty} T\varphi_n$ exists, then $\bar{T}\varphi = \lim_{n\to\infty} T\varphi_n$.

(d) A linear operator S is said to be an *extension* of T if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $S\varphi = T\varphi$ for all $\varphi \in \mathcal{D}(T)$. We then write $T \subset S$.

Lemma 2 *If $A : \varphi(x) \in \mathcal{D}(A) \subset L^2(M, \mu) \mapsto a(x)\varphi(x) \in L^2(M, \mu)$ is a multiplication operator, then A is closable and its closure is again multiplication by $a(x)$.*

Example 3 It is easy to construct, using an algebraic basis, a linear operator whose domain is the entire Hilbert space, but which is unbounded. (We are of course assuming that the Hilbert space is infinite dimensional.) By the closed graph theorem, this operator cannot be closed. So it provides an extreme example of an operator whichis not closable.

Example 4 On the other hand, it is also possible for an operator to have many closed extensions. Here is an example. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$ and the operator is

$$\mathcal{D}(A) = \left\{ f \in C_0^\infty(\mathbb{R}) \ \middle|\ \int_{-\infty}^{\infty} f(k)\ dk = \int_{-\infty}^{\infty} kf(k)\ dk = 0 \right\} \quad (Af)(k) = (1+k^2)f(k)$$

If you take Fourier tranforms, this operator becomes the differential operator $-\frac{d^2}{dx^2} + 1$ with "initial conditions" $\hat{f}(0) = \frac{d\hat{f}}{dx}(0) = 0$. Set

$$p_0(k) = \tfrac{1}{1+k^2} \qquad p_1(k) = \tfrac{k}{1+k^2}$$

The the closure of A is

$$\mathcal{D}(\bar{A}) = \left\{ f \in L^2(\mathbb{R}) \ \middle|\ (1+k^2)f \in L^2(\mathbb{R}), (1+k^2)f \perp p_0, p_1 \right\} \quad (\bar{A}f)(k) = (1+k^2)f(k)$$

Choose any nonzero $p, q \in \operatorname{span}\{p_0, p_1\}$ and a nonzero $p^\perp \in \operatorname{span}\{p_0, p_1\}$ which is perpendicular to p. The following are all closed extensions of A.

$$\begin{aligned}
&\mathcal{D}(A_1) = \left\{ f \in L^2(\mathbb{R}) \ \middle|\ (1+k^2)f \in L^2(\mathbb{R}), (1+k^2)f \perp p \right\} && (A_1 f)(k) = (1+k^2)f(k) \\
&\mathcal{D}(A_2) = \left\{ f \in L^2(\mathbb{R}) \ \middle|\ (1+k^2)f \in L^2(\mathbb{R}) \right\} && (A_2 f)(k) = (1+k^2)f(k) \\
&\mathcal{D}(A_3) = \mathcal{D}(A_1) = \left\{ \alpha \tfrac{p^\perp}{1+k^2} + f \ \middle|\ \alpha \in \mathbb{C},\ (1+k^2)f \in \{p_0, p_1\}^\perp \right\} \\
& && A_3\big(\alpha \tfrac{p^\perp}{1+k^2} + f\big) = \alpha q + (1+k^2)f
\end{aligned}$$

Definition 5 Let T be a densely defined linear operator on a Hilbert space $\mathcal{H}$. The *adjoint* T^* of T is defined by

$$\begin{aligned}
\varphi \in \mathcal{D}(T^*) &\iff \exists\ \eta \in \mathcal{H} \text{ s.t. } \langle T\psi, \varphi\rangle_{\mathcal{H}} = \langle \psi, \eta\rangle_{\mathcal{H}}\ \ \forall\ \psi \in \mathcal{D}(T) \\
T^*\varphi = \eta &\iff \langle T\psi, \varphi\rangle_{\mathcal{H}} = \langle \psi, \eta\rangle_{\mathcal{H}}\ \ \forall\ \psi \in \mathcal{D}(T)
\end{aligned}$$

Remark 6

(a) The η of the last definition is unique when it exists, because $\mathcal{D}(T)$ is dense.

(b) By the Riesz representation theorem, $\varphi \in \mathcal{D}(T^*)$ if and only if there is a constant K_φ such that $\big|\langle T\psi, \varphi\rangle_{\mathcal{H}}\big| \le K_\varphi \|\psi\|_{\mathcal{H}}$ for all $\psi \in \mathcal{D}(T)$.

(c) $\mathbf{0} \in \mathcal{D}(T^*)$ always.

(d) If $S \subset T$, then $T^* \subset S^*$.

Example 7 The following example shows that it is possible to have $\mathcal{D}(T^*) = \{\mathbf{0}\}$. Let

- ∘ $\mathcal{H} = L^2(\mathbb{R})$,
- ∘ $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and
- ∘ for each $k \in \mathbb{N}$, $f_k(x) = e^{ikx}$. Note that $f_k \notin \mathcal{H}$.

We define the domain of T to be $\mathcal{D}(T) = \mathcal{B}_0(\mathbb{R})$, the set of all bounded Borel functions on $\mathbb{R}$ that are of compact support. This domain is dense in $L^2(\mathbb{R})$. For $\varphi \in \mathcal{D}(T)$,

$$T\varphi = \sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} f_n(x)\varphi(x)\ dx \right] \mathbf{e}_n$$

Step 1: We first check that T is well–defined. Let $\varphi \in \mathcal{D}(T)$. Then there is some integer m such that $\varphi(x)$ vanishes outside of $[-m\pi, m\pi]$. Then, for each $k \in \mathbb{Z}$,

$$\int_{-\infty}^{\infty} f_k(x)\varphi(x)\ dx = \int_{-m\pi}^{m\pi} f_k(x)\varphi(x)\ dx = \int_{-\pi}^{\pi} e^{ikmt}\ m\varphi(mt)\ dt$$

is the km^{th} Fourier coefficient of the function $m\varphi(mt)$. Since the sum of the squares of all of the Fourier coefficents is, up to a factor of 2π, the L^2 norm of $m\varphi(mt)$, which is finite, T is well–defined.

Step 2: We now check that $\mathcal{D}(T^*) = \{\mathbf{0}\}$. Let $\psi \in \mathcal{D}(T^*)$ and $\varphi \in \mathcal{D}(T) = \mathcal{B}_0(\mathbb{R})$. Pick an $m \in \mathbb{N}$ with $\varphi(x)$ vanishing except for x in $[-m\pi, m\pi]$. Then

$$\langle \varphi, T^*\psi \rangle = \langle T\varphi, \psi \rangle = \sum_{n=1}^{\infty} \left[\int_{-\infty}^{\infty} \overline{f_n(x)\varphi(x)}\ dx \right]\ \langle \mathbf{e}_n, \psi \rangle = \sum_{n=1}^{\infty} \Big\langle \varphi\,,\ \langle \mathbf{e}_n, \psi \rangle\, \overline{f_n} \Big\rangle$$

Since

$$\int_{-m\pi}^{m\pi} \overline{f_k(x)} f_\ell(x)\ dx = \int_{-m\pi}^{m\pi} e^{i(\ell-k)x}\ dx = \begin{cases} 2m\pi & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

The series $\sum_{n=1}^{\infty} \langle \mathbf{e}_n, \psi \rangle\, \overline{f_n}$ converges in $\mathcal{H}$ and

$$\langle \varphi, T^*\psi \rangle = \Big\langle \varphi\,,\ \sum_{n=1}^{\infty} \langle \mathbf{e}_n, \psi \rangle\, \overline{f_n} \Big\rangle$$

This is true for all bounded Borel functions φ supported in $[-m\pi, m\pi]$ so that

$$T^*\psi = \sum_{n=1}^{\infty} \langle \mathbf{e}_n, \psi \rangle\, \overline{f_n} \qquad \text{a.e. on } [-m\pi, m\pi]$$

and

$$\infty > \big\| T^*\psi \big\|_{L^2(\mathbb{R})} \geq \big\| T^*\psi \big\|_{L^2([-m\pi, m\pi])} = \sum_{n=1}^{\infty} \big| \langle \mathbf{e}_n, \psi \rangle \big|^2 (2m\pi)$$

Since this is true for all m, we must have $\langle \mathbf{e}_n, \psi \rangle = 0$ for all $n \in \mathbb{N}$ and hence $\psi = 0$.

Theorem 8 *Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator. Then*
(a) T^* is closed.
(b) T is closable if and only if $\mathcal{D}(T^*)$ is dense. If so, $\bar{T} = T^{**}$.
(c) If T is closable, then $(\bar{T})^* = T^*$.

Definition 9 Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator.
(a) T is called *symmetric* if

$$\langle \varphi, T\psi \rangle = \langle T\psi, \varphi \rangle \qquad \text{for all } \varphi, \psi \in \mathcal{D}(T)$$

That is, if $T \subset T^*$.
(b) T is called *self–adjoint* if $T = T^*$.

Remark 10

(a) If T is self–adjoint, then it is automatically closed.

(b) Let (M, μ) be a measure space. If $f(m)$ is a real–valued, measurable function on M, then the operator of multiplication by f, with domain $\{ \psi \in L^2(M, \mu) \mid f\psi \in L^2(M, \mu) \}$ is self–adjoint.

Theorem 11 *Let $\mathcal{H}$ be a Hilbert space. A densely defined, symmetric operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ is self–adjoint if and only if the ranges of $\operatorname{ran}(T - i\mathbb{1}) = \operatorname{ran}(T + i\mathbb{1}) = \mathcal{H}$.*

Definition 12 Let T be a densely defined, closed operator on a Hilbert space $\mathcal{H}$.

(a) The *resolvent set*, $\rho(T)$, of T is the set of all complex numbers λ such that $\lambda\mathbb{1} - T$ is a bijection (between $\mathcal{D}(T)$ and $\mathcal{H}$) with bounded inverse.

(b) The *resolvent* of T at $\lambda \in \rho(T) \subset \mathbb{C}$ is $R_\lambda(T) = \big(\lambda\mathbb{1} - T\big)^{-1}$.

(c) The *spectrum* of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

(d) The complex number λ is in the *point spectrum*, $\sigma_p(T)$, of T if $\lambda\mathbb{1} - T$ is not injective. That is, if there is a nonzero vector $\mathbf{x} \in \mathcal{H}$ such that $T\mathbf{x} = \lambda\mathbf{x}$. Then $\mathbf{x}$ is said to be an eigenvector of T with eigenvalue λ.

(e) The complex number λ is in the *residual spectrum*, $\sigma_r(T)$, of T if $\lambda\mathbb{1} - T$ is injective but the range of T is not dense in $\mathcal{B}$.

Example 13 Here is an example which shows, firstly, that an ubounded operator T may have $\sigma(T) = \emptyset$ and secondly that "just changing the domain of an operator" can change its spectrum. The Hlbert space $\mathcal{H} = L^2([0,1])$. Denote by AC$[0,1]$ the set of all functions $f : [0,1] \to \mathbb{C}$ of the form $f(x) = C + \int_0^x \varphi(t)\, dt$ with $C \in \mathbb{C}$ and $\varphi \in L^2([0,1])$. For such a function, define $\frac{df}{dx}(x) = \varphi(x)$. All of the operators in this example will be closed operators with domain contained in AC$[0,1]$ and all operators will have the action $f(x) \mapsto i\frac{df}{dx}(x)$. Here are the examples.

(a) If $\mathcal{D}(T) = \mathrm{AC}[0,1]$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \mathbb{C}$. In fact $\sigma_p(T) = \mathbb{C}$, since, for any $\lambda \in \mathbb{C}$, $e^{-i\lambda x}$ is an eigenfunction for T with eigenvalue λ.

(b) If $\mathcal{D}(T) = \{\ f \in \mathrm{AC}[0,1]\ \big|\ f(0) = 0\ \}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \emptyset$. In fact, for any $\lambda \in \mathbb{C}$, the resolvent operator $(\lambda\mathbb{1} - T)^{-1}$ is

$$\big(R_\lambda(T)\psi\big)(x) = i\int_0^x e^{-i\lambda(x-t)}\psi(t)\, dt$$

(c) Let $\alpha \in \mathbb{C}$ be nonzero. If $\mathcal{D}(T) = \{\ f \in \mathrm{AC}[0,1]\ \big|\ f(0) = \alpha f(1)\ \}$ with $Tf = i\frac{df}{dx}$, then $\sigma(T) = \{\ -i\ln\alpha + 2k\pi\ \big|\ k \in \mathbb{Z}\ \}$. Again the spectrum consists solely of eigenvalues. If $\lambda = -i\ln\alpha + 2k\pi$ for some $k \in \mathbb{Z}$, then $e^{-i\lambda x}$ is an eigenfunction for T with eigenvalue λ. For λ not of the form $-i\ln\alpha + 2k\pi$ for all $k \in \mathbb{Z}$, the resolvent operator $(\lambda\mathbb{1} - T)^{-1}$ is

$$\big(R_\lambda(T)\psi\big)(x) = \int_0^1 G_\lambda(x,t)\psi(t)\, dt$$

with

$$G_\lambda(x,t) = \begin{cases} \frac{i\alpha e^{i\lambda(t-x-1)}}{1-\alpha e^{-i\lambda}} & \text{if } x < t \\ \frac{ie^{i\lambda(t-x)}}{1-\alpha e^{-i\lambda}} & \text{if } x > t \end{cases}$$

Theorem 14 (Cayley Transform) *Let $\mathcal{H}$ be a Hilbert space.*

(a) If $A = A^*$ is a self–adjoint operator on $\mathcal{H}$, then $C_+(A) = (A - i\mathbb{1})(A + i\mathbb{1})^{-1}$ exists and is a unitary operator on $\mathcal{H}$. Furthermore $1 \notin \sigma_p\big(C_+(A)\big)$.

(b) If U is a unitary operator on $\mathcal{H}$ and $1 \notin \sigma_p(U)$, then $C_-(U) = i(\mathbb{1}+U)(\mathbb{1}-U)^{-1}$ exists and is self–adjoint.

(c) C_+ and C_- are inverses. That is, $C_+\big(C_-(U)\big) = U$ and $C_-\big(C_+(A)\big) = A$ for all unitary U with $1 \notin \sigma_p(U)$ and all self–adjoint A.

(d) If T is a symmetric operator with domain $D(T)$, then $C_+(T) = (T - i\mathbb{1})(T + i\mathbb{1})^{-1}$ is a well–defined isometric operator with domain $R(T + i\mathbb{1})$ and range $R(T - i\mathbb{1})$. Furthermore $1 \notin \sigma_p\big(C_+(T)\big)$ and $T = C_-\big(C_+(T)\big)$.

Remark 15 Let T be a symmetric operator. By the last theorem, there is a one–to–one correspondance between self–adjoint extensions of T and unitary operators U that

(a) extend the isometric operator $C_+(T)$ and

(b) obey $1 \notin \sigma_p(U)$.

First forget about the condition that $1 \notin \sigma_p(U)$. It's a simple matter to construct all unitary operators that extend $C_+(T)$. Since the closure $\overline{C_+(T)}$ is an isometric operator with domain $\overline{R(T + i\mathbb{1})}$ and range $\overline{R(T - i\mathbb{1})}$, there is a one–to–one–correspondance between such unitary extensions and unitary maps $\tilde{U} : R(T + i\mathbb{1})^\perp \to R(T - i\mathbb{1})^\perp$. Now back to condition (b). The following lemma shows it is automatically satisfied!

Lemma 16 *Let $\mathcal{H}$ be a Hilbert space and $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined, symmetric operator. Let U be a unitary extension of $C_+(T)$. Then $1 \notin \sigma_p(U)$.*

Definition 17 An operator is called *essentially self–adjoint* if its closure is self–adjoint. A subset $\mathcal{D} \subset \mathcal{H}$ is called a *core* for the self–adjoint operator T if $\overline{T \restriction \mathcal{D}} = T$.

Theorem 18 *Let T be a densely defined operator on a Hilbert space $\mathcal{H}$. Then*

$$T \text{ is essentially self–adjoint} \iff T \text{ has a unique self–adjoint extension}$$

Theorem 19 *Let T be a symmetric operator on a Hilbert space $\mathcal{H}$.*

(a) If $\dim (T + i)^\perp = \dim (T - i)^\perp = 0$, then T has a unique self–adjoint extension, and so is essentially self–adjoint.

(b) If $\dim (T + i)^\perp = \dim (T - i)^\perp \geq 1$, then T has infinitely many distinct self–adjoint extensions.

(c) If $\dim (T + i)^\perp \neq \dim (T - i)^\perp$, then T has no self–adjoint extensions.

Theorem 20 (Spectral Theorem - Multiplication Operator Version)
Let A be a self–adjoint operator on a Hilbert space $\mathcal{H}$. There exist

- *a measure space $\langle M, \mu \rangle$,*
- *a measurable function $a : M \to \mathbb{R}$, and*
- *a unitary operator $U : \mathcal{H} \to L^2(M, \mu)$*

such that UAU^{-1} is the operator of multiplication by the function $a(m)$, with domain $\{ \varphi \in L^2(M, \mu) \mid a\varphi \in L^2(M, \mu) \}$. If $\mathcal{H}$ is separable, μ can be chosen to be a finite measure.

Definition 21 (Projection Valued Measure)

(a) Denote by $\mathcal{B}_{\mathbb{R}}$ the σ–algebra of Borel subsets of $\mathbb{R}$ and by $\mathcal{L}(\mathcal{H})$ the set of a bounded operators on $\mathcal{H}$. A projection valued measure is a map $E : \mathcal{B}_{\mathbb{R}} \to \mathcal{L}(\mathcal{H})$ that obeys the following conditions.

(i) For each $B \in \mathcal{B}_{\mathbb{R}}$, the operator $E(B)$ is an orthogonal projection on some closed subspace of $\mathcal{H}$. That is, $E(B)^2 = E(B)$ and $E(B) = E(B)^*$.

(ii) $E(\emptyset) = 0$ and $E(\mathbb{R}) = \mathbb{1}$

(iii) If $\{B_n\}_{n\in\mathbb{N}}$ is a countable family of disjoint Borel subsets of $\mathbb{R}$, then

$$E\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \operatorname*{s-lim}_{N\to\infty} \sum_{n=1}^{N} E(B_n)$$

(iv) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$. (This condition is a consequence of (i) and (iii).)

(b) Given a projection valued measure E and a Borel function g (not necessarily bounded), the operator $\int g(\lambda)\ dE(\lambda)$ is defined by

$$\mathcal{D}\Big(\int g(\lambda)\ dE(\lambda)\Big) = \Big\{\psi \in \mathcal{H} \Big| \int |g(\lambda)|^2\ d\,\langle \psi, E(\lambda)\psi\rangle < \infty\Big\}$$

$$\Big\langle \psi, \int g(\lambda)\ dE(\lambda)\ \varphi\Big\rangle = \int g(\lambda)\ d\,\langle \psi, E(\lambda)\varphi\rangle$$

Theorem 22 (Spectral Theorem - Projection–valued Measure Version)
There is a 1–1 correspondence between self–adjoint operators and projection–valued measures $A \leftrightarrow E_A$ such that

$$A = \int \lambda\ dE_A(\lambda)$$