

Review of Spectral Theory

Definition 1 Let \mathcal{H} be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$.

- (a) A is called *self-adjoint* if $A = A^*$.
- (b) A is called *unitary* if $A^*A = AA^* = \mathbb{1}$. Equivalently, A is unitary if it is bijective (i.e. 1-1 and onto) and preserves inner products.
- (c) A is called *normal* if $A^*A = AA^*$. That is, if A commutes with its adjoint.
- (d) Let \mathcal{X} and \mathcal{Y} be Banach spaces. A linear operator $C : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be compact if for each bounded sequence $\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{X}$, there is a subsequence of $\{Cx_i\}_{i \in \mathbb{N}}$ that is convergent.

Remark 2 Let \mathcal{H} be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. In this remark, we assume that \mathcal{H} is finite dimensional, so that A is multiplication by a matrix, that we also denote A .

(a) If A is self-adjoint, we have:

- (i) All eigenvalues of A are real, since

$$\mathbf{0} \neq \mathbf{x} \in \mathcal{H}, A\mathbf{x} = \lambda\mathbf{x} \implies \lambda \langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{x}, A\mathbf{x} \rangle = \langle A\mathbf{x}, \mathbf{x} \rangle = \bar{\lambda} \langle \mathbf{x}, \mathbf{x} \rangle$$

- (ii) Eigenvectors of A that correspond to different eigenvalues are perpendicular, since

$$\begin{aligned} \mathbf{0} \neq \mathbf{x} \in \mathcal{H}, A\mathbf{x} = \lambda\mathbf{x}, \quad \mathbf{0} \neq \mathbf{y} \in \mathcal{H}, A\mathbf{y} = \mu\mathbf{y}, \quad \lambda \neq \mu \\ \implies \lambda \langle \mathbf{x}, \mathbf{y} \rangle = \langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{y} \rangle = \mu \langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

- (iii) A is diagonalizable. That is, there is a basis of \mathcal{H} consisting of eigenvectors of A .

See the notes “Families of Commuting Normal Matrices”.

As a consequence of (ii) and (iii), there is an orthonormal basis of \mathcal{H} consisting of eigenvectors of A . Call the basis vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Write $U = [\mathbf{x}_1 \ \dots \ \mathbf{x}_n]$ (that is, the columns of U are eigenvectors of A) and denote by D the diagonal matrix whose diagonal entries are the λ_j 's. Then U is unitary (the condition that $U^*U = \mathbb{1}$ is the same as the condition that the columns are orthonormal) and

$$AU = [A\mathbf{x}_1 \ \dots \ A\mathbf{x}_n] = [\lambda\mathbf{x}_1 \ \dots \ \lambda_n\mathbf{x}_n] = UD \quad \text{or} \quad U^{-1}AU = D \quad \text{or} \quad U^*AU = D$$

(b) This and similar arguments give that

A is self-adjoint $\iff \mathcal{H}$ has an orthonormal basis of eigenvectors of A and
all eigenvalues of A are real

A is unitary $\iff \mathcal{H}$ has an orthonormal basis of eigenvectors of A and
all eigenvalues λ of A obey $|\lambda| = 1$

A is normal $\iff \mathcal{H}$ has an orthonormal basis of eigenvectors of A

Remark 3

(a) When \mathcal{H} is finite dimensional, A is normal if and only if there is a unitary matrix U and a diagonal matrix D such that $U^*AU = D$. That is, A is diagonalizable by a unitary matrix. The corresponding statement when \mathcal{H} is infinite dimensional, is that A is normal if and only if there is a unitary operator U such that U^*AU is a multiplication operator. We shall prove this.

(b) If an operator is either self-adjoint or unitary, it is also normal.

(c) An operator A is normal if and only if it can be written in the form $A = B + iC$ with B and C self-adjoint and commuting. (Take $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$.)

Remark 4 In quantum mechanics, “physical observables” tend to be self-adjoint operators — energy, momentum, etc are real quantities. In quantum mechanics, time evolution is by a unitary operator. “Total probability is preserved.”

Remark 5 When \mathcal{H} is finite dimensional, all operators are compact. Even when \mathcal{H} is infinite dimensional, compact operators behave a lot like finite dimensional matrices. See Theorem 18. They tend to be easier to work with than other operators.

Remark 6 In the finite dimensional case, we say that A is diagonalizable if there is an invertible (but not necessarily unitary) matrix V such that VAV^{-1} is diagonal. Then \mathcal{H} has a basis of eigenvectors of A , but the basis vectors need not be mutually perpendicular.

A natural extension of this finite dimensional definition to infinite dimensions would be that A is diagonalizable if there is a bounded linear bijection $V : \mathcal{H} \rightarrow \mathcal{H}' = L^2(\mathcal{M}, \mu)$ (then it has a bounded inverse by the inverse mapping theorem) such that VAV^{-1} is a multiplication operator. Again, V need not be unitary. But we can make V unitary by changing the inner product on \mathcal{H} (thereby changing the meaning of “orthogonal”). Define

$$\langle \mathbf{x}, \mathbf{y} \rangle'_{\mathcal{H}} = \langle V\mathbf{x}, V\mathbf{y} \rangle_{\mathcal{H}'}$$

Then V is unitary as a map from $(\mathcal{H}, \langle \cdot, \cdot \rangle'_{\mathcal{H}})$ to \mathcal{H}' . Here $(\mathcal{H}, \langle \cdot, \cdot \rangle'_{\mathcal{H}})$ is the vector space \mathcal{H} equipped with the inner product $\langle \cdot, \cdot \rangle'_{\mathcal{H}}$ instead of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Changing the inner product in this way changes the lengths of vectors in \mathcal{H} and also the angles between vectors in \mathcal{H} , but does not change the topology since

$$\|V^{-1}\|^{-1} \|\mathbf{x}\|_{\mathcal{H}} \leq \|\mathbf{x}\|'_{\mathcal{H}} \leq \|V\| \|\mathbf{x}\|_{\mathcal{H}}$$

As A is diagonalizable by a unitary operator if and only if it is normal, we have that A is diagonalizable if and only if A is normal with respect to some inner product on \mathcal{H} that

gives a topology equivalent to the original topology. (All such inner products are of the form $\langle V\mathbf{x}, V\mathbf{y} \rangle_{\mathcal{H}}$ for some bijection V on \mathcal{H} that is bounded with bounded inverse. See Corollary 35 in the notes “Review of Hilbert and Banach Spaces”.)

Definition 7 Let \mathcal{B} be a Banach space and $T \in \mathcal{L}(\mathcal{B})$.

(a) The *resolvent set*, $\rho(T)$, of T is the set of all complex numbers λ such that $\lambda\mathbb{1} - T$ is a bijection with bounded inverse.

(b) The *resolvent* of T at $\lambda \in \rho(T) \subset \mathbb{C}$ is $R_\lambda(T) = (\lambda\mathbb{1} - T)^{-1}$.

(c) The *spectrum* of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

(d) The complex number λ is in the *point spectrum*, $\sigma_p(T)$, of T if $\lambda\mathbb{1} - T$ is not injective. That is, if there is a nonzero vector $\mathbf{x} \in \mathcal{H}$ such that $T\mathbf{x} = \lambda\mathbf{x}$. Then \mathbf{x} is said to be an eigenvector of T with eigenvalue λ .

(e) The complex number λ is in the *residual spectrum*, $\sigma_r(T)$, of T if $\lambda\mathbb{1} - T$ is injective but the range of T is not dense in \mathcal{B} .

Remark 8 Let M be an $n \times n$ matrix. By definition, M is diagonalizable if there is an invertible matrix V and a diagonal matrix D such that $V^{-1}MV = D$. In this case the columns of V are eigenvectors of M that form a basis for \mathbb{C}^n , the diagonal elements of D are the eigenvalues of M and $\sigma(M) = \sigma_p(M)$ is the set of all eigenvalues of M .

If M is not diagonalizable, there still exists an invertible matrix V such that $V^{-1}MV$ is in Jordan form. This means that it is of the form

$$\begin{bmatrix} J_1 & 0 & \cdots & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & J_m & \end{bmatrix}$$

with each diagonal block being a Jordan block. A 4×4 Jordan block is of the form

$$B_{\lambda,4} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Observe that $\lambda\mathbb{1} - B_{\lambda,4}$ has range $\{ (z_1, z_2, z_3, 0) \mid z_1, z_2, z_3 \in \mathbb{C} \}$ which is not dense. In this finite dimensional world, $\lambda\mathbb{1} - B_{\lambda,4}$ must also have a kernel (since the dimension of the range plus the dimension of the kernel must be the dimension of the world). That is, λ

must be an eigenvalue. For an infinite dimensional world that is no longer the case. The existence of residual spectrum signals the failure of diagonalizability.

Example 9

Multiplication Operators: Let

- (X, \mathcal{M}, μ) be a semifinite measure space,
- $1 \leq p \leq \infty$ and
- $a : X \rightarrow \mathbb{C}$ be a bounded measurable function on X .

Define the bounded operator $A : L^p(X, \mathcal{M}, \mu) \rightarrow L^p(X, \mathcal{M}, \mu)$ by

$$(A\varphi)(x) = a(x)\varphi(x)$$

Then

$$\begin{aligned} \rho(A) &= \{ \lambda \in \mathbb{C} \mid \exists \varepsilon > 0 \text{ such that } |\lambda - a(x)| \geq \varepsilon \text{ a.e.} \} \\ \sigma_p(A) &= \{ \lambda \in \mathbb{C} \mid \mu(\{x \in X \mid a(x) = \lambda\}) > 0 \} \\ \sigma_r(A) &= \begin{cases} \emptyset & \text{if } 1 \leq p < \infty \\ \{ \lambda \in \mathbb{C} \mid \nexists \varepsilon > 0 \text{ such that } |\lambda - a(x)| \geq \varepsilon \text{ a.e.} \} \setminus \sigma_p(A) & \text{if } p = \infty \end{cases} \end{aligned}$$

Shift Operators: Define the right and left shift operators acting on ℓ^2 by

$$\begin{aligned} L(\alpha_1, \alpha_2, \alpha_3, \dots) &= (\alpha_2, \alpha_3, \dots) \\ R(\alpha_1, \alpha_2, \alpha_3, \dots) &= (0, \alpha_1, \alpha_2, \alpha_3, \dots) \end{aligned}$$

Then

$$\begin{aligned} \rho(L) &= \{ \lambda \in \mathbb{C} \mid |\lambda| > 1 \} & \rho(R) &= \{ \lambda \in \mathbb{C} \mid |\lambda| > 1 \} \\ \sigma_p(L) &= \{ \lambda \in \mathbb{C} \mid |\lambda| < 1 \} & \sigma_p(R) &= \emptyset \\ \sigma_r(L) &= \emptyset & \sigma_r(R) &= \{ \lambda \in \mathbb{C} \mid |\lambda| < 1 \} \end{aligned}$$

See the notes “Spectral Theory Examples” for derivations and other examples.

Definition 10 Let \mathcal{X} be a Banach space and \mathcal{D} an open subset of \mathbb{C} . A function $\mathbf{x} : \mathcal{D} \rightarrow \mathcal{X}$ is analytic at $z_0 \in \mathcal{D}$ if

$$\lim_{z \rightarrow z_0} \frac{\mathbf{x}(z) - \mathbf{x}(z_0)}{z - z_0}$$

exists.

Lemma 11 *Let B be a bounded linear operator on the Banach space \mathcal{B} . Assume that B has a bounded inverse and that C is a bounded operator on \mathcal{B} with $\|C\| < \|B^{-1}\|^{-1}$. Then $B + C$ is 1-1 and onto and has a bounded inverse and furthermore*

$$\|(B + C)^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \|B^{-1}\|\|C\|} \quad \|(B + C)^{-1} - B^{-1}\| \leq \frac{\|B^{-1}\|^2\|C\|}{1 - \|B^{-1}\|\|C\|}$$

Theorem 12 *Let \mathcal{B} be a Banach space and $T \in \mathcal{L}(\mathcal{B})$. Then*

(a) $\rho(T)$ is an open subset of \mathbb{C} . Furthermore, if $\lambda \in \rho(T)$, then

$$\lim_{\mu \rightarrow \lambda} \|R_\mu(T) - R_\lambda(T)\| = 0$$

(b) $R_\lambda(T)$ is an analytic $\mathcal{L}(\mathcal{B})$ -valued function of λ on $\rho(T)$.

(c) First resolvent formula: If $\lambda, \mu \in \rho(T)$, then $R_\lambda(T)$ and $R_\mu(T)$ commute and

$$R_\lambda(T) - R_\mu(T) = (\mu - \lambda)R_\mu(T)R_\lambda(T)$$

Second resolvent formula: If $S \in \mathcal{L}(\mathcal{B})$ and $\lambda \in \rho(S) \cap \rho(T)$, then

$$R_\lambda(S) - R_\lambda(T) = R_\lambda(S)(S - T)R_\lambda(T) = R_\lambda(T)(S - T)R_\lambda(S)$$

(d) If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$.

(e) $\sigma(T) \neq \emptyset$.

Lemma 13 *Let B be a bounded linear operator on the Banach space \mathcal{B} . If $P(z)$ is a polynomial and $\lambda \in \sigma(B)$ then $P(\lambda) \in \sigma(P(B))$.*

Theorem 14 *Let \mathcal{B} be a Banach space and $T \in \mathcal{L}(\mathcal{B})$. The spectral radius of T is defined to be*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

We have

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}$$

If \mathcal{B} is a Hilbert space and T is self-adjoint, then

$$r(T) = \|T\|$$

Theorem 15 Let \mathcal{H} be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$.

- (a) $\sigma(A^*) = \overline{\sigma(A)}$.
- (b) $\lambda \in \sigma_r(A) \implies \bar{\lambda} \in \sigma_p(A^*)$
 $\lambda \in \sigma_p(A) \implies \bar{\lambda} \in \sigma_p(A^*) \cup \sigma_r(A^*)$

Lemma 16 Let \mathcal{H} be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be normal.

- (a) If φ is an eigenvector of A of eigenvalue λ , then φ is an eigenvector of A^* of eigenvalue $\bar{\lambda}$.
- (b) Eigenvectors of A with different eigenvalues are orthogonal.
- (c) A has no residual spectrum.
- (d) If $A = A^*$, then $\sigma(A) \subset \mathbb{R}$.
- (e) $\lambda \in \sigma(A)$ if and only if, for each $\varepsilon > 0$, there exists a $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$ and $\|(\lambda\mathbb{1} - A)\varphi\| < \varepsilon$.

Theorem 17 Let \mathcal{H} and \mathcal{H}' be Hilbert spaces. Let $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ be bijective and $A \in \mathcal{L}(\mathcal{H})$ and set $A' = UAU^{-1} \in \mathcal{L}(\mathcal{H}')$. Then

$$\rho(A') = \rho(A) \quad \sigma(A') = \sigma(A) \quad \sigma_p(A') = \sigma_p(A) \quad \sigma_r(A') = \sigma_r(A)$$

Theorem 18 (The Spectrum of Compact Operators) Let $C : \mathcal{X} \rightarrow \mathcal{X}$ be a compact operator on the Banach space \mathcal{X} . The spectrum of C consists of at most countably many points. For any $\varepsilon > 0$, $\{ \lambda \in \sigma(C) \mid |\lambda| > \varepsilon \}$ is finite. If $0 \neq \lambda \in \sigma(C)$, then λ is an eigenvalue of C of finite multiplicity.

Theorem 19 (Spectral Theorem - Multiplication Operator Version)

Let A be a bounded self-adjoint operator on a Hilbert space \mathcal{H} . There exist

- a measure space $\langle M, \Sigma, \mu \rangle$,
- a bounded measurable function $a : M \rightarrow \mathbb{R}$, and
- a unitary operator $U : \mathcal{H} \rightarrow L^2(M, \Sigma, \mu)$

such that

$$(UAU^{-1}\varphi)(m) = a(m)\varphi(m)$$

for all $\varphi \in L^2(M, \Sigma, \mu)$. If \mathcal{H} is separable, μ can be chosen to be a finite measure.

Example 20 Let A be a self-adjoint, compact operator and let $\{\varphi_n\}_{n \in \mathcal{I}}$ be a complete orthonormal basis for \mathcal{H} consisting of eigenvectors of A . Denote by λ_n the eigenvalue of A for the eigenvector φ_n .

Think of $\ell^2(\mathcal{I})$ as L^2 of the measure space \mathcal{I} , equipped with the counting measure. So, think of an element of $\ell^2(\mathcal{I})$ as a function on \mathcal{I} rather than a sequence. Define the unitary operator $U : \mathcal{H} \rightarrow \ell^2(\mathcal{I})$ by

$$\left(U \left(\sum_{n \in \mathcal{I}} x_n \varphi_n \right) \right)(m) = x_m$$

The inverse operator $U^{-1} = U^* : \ell^2(\mathcal{I}) \rightarrow \mathcal{H}$ is given by

$$U^{-1}v = \sum_{n \in \mathcal{I}} v(n) \varphi_n$$

For each $m \in \mathbb{N}$, denote by e_m the element of $\ell^2(\mathcal{I})$ all of whose components are zero except for the m^{th} , which is 1. That is

$$e_m(n) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Observe that $U\varphi_m = e_m$, for each $m \in \mathcal{I}$. Also define the function $a : \mathbb{N} \rightarrow \mathbb{R}$ by $a(m) = \lambda_m$. Then, for each $v \in \ell^2(\mathcal{I})$ and each $m \in \mathcal{I}$,

$$\begin{aligned} (UAU^{-1}v)(m) &= \langle e_m, UAU^{-1}v \rangle_{\ell_2} = \langle U^{-1}e_m, AU^{-1}v \rangle_{\mathcal{H}} = \langle \varphi_m, AU^{-1}v \rangle_{\mathcal{H}} \\ &= \langle A\varphi_m, U^{-1}v \rangle_{\mathcal{H}} = \lambda_m \langle \varphi_m, U^{-1}v \rangle_{\mathcal{H}} = \lambda_m \langle U\varphi_m, v \rangle_{\ell_2} \\ &= \lambda_m \langle e_m, v \rangle_{\ell_2} = a(m)v(m) \end{aligned}$$

If, as will often be the case, $\mathcal{I} = \mathbb{N}$, the counting measure on \mathcal{I} is not finite. But it is easy to rework the above construction so as to use a finite measure space. Define the measure μ on \mathbb{N} by $\mu(\{m\}) = \frac{1}{2^m}$. Then define $U : \mathcal{H} \rightarrow L^2(\mathbb{N}, \mu)$ by $U\left(\sum_{n \in \mathcal{I}} x_n \varphi_n\right)(m) = 2^{m/2}x_m$ and the function a by $a(m) = \lambda_m$ again.

Theorem 21 (Spectral Theorem - Multiplication Operator, with Multiplicity)

Let A be a bounded self-adjoint operator on a separable Hilbert space \mathcal{H} . There exist $N \in \mathbb{N} \cup \{\infty\}$ and measures μ_n , $1 \leq n \leq N$, on the spectrum $\sigma(A) \subset \mathbb{R}$, of A , and a unitary operator $U : \mathcal{H} \rightarrow \bigoplus_{n=1}^N L^2(\sigma(A), \mu_n)$ such that

$$(UAU^{-1}\varphi)_n(\lambda) = \lambda \varphi_n(\lambda)$$

for all $\varphi \in \bigoplus_{n=1}^N L^2(\sigma(A), \mu_n)$.

Example 22 Let A be a self-adjoint, compact operator and let $\{\varphi_n\}_{n \in \mathcal{I}}$ be a complete orthonormal basis for \mathcal{H} consisting of eigenvectors of A . Denote by λ_n the eigenvalue of A for the eigenvector φ_n . Denote by $1 \leq N \leq \infty$ the supremum of the multiplicities of the eigenvalues of A . Define, for each $1 \leq n \leq N$, the Borel measure μ_n on \mathbb{R} by

$$\mu_n(B) = \sum_{n \in \mathcal{I}} \begin{cases} 1 & \text{if } \lambda_n \in B \text{ and } \lambda_n \text{ has multiplicity at least } n \\ 0 & \text{otherwise} \end{cases}$$

Then $\oplus_{n=1}^N L^2(\mathbb{R}, \mu_n)$ is the set of all N -vector valued functions on \mathbb{R} with

$$\|\vec{\psi}\|^2 = \sum_{n=1}^N \int d\mu_n(x) |\psi_n(x)|^2$$

Define the unitary operator $U\mathcal{H} \rightarrow \oplus_{n=1}^N L^2(\mathbb{R}, \mu_n)$ by

$$U \left(\sum_{m=1}^{\infty} x_m \varphi_m \right)_n (\lambda) = \sum_{m=1}^{\infty} x_m \begin{cases} 1 & \text{if } \varphi_m \text{ is the } n^{\text{th}} \text{ eigenvector of eigenvalue } \lambda \\ 0 & \text{otherwise} \end{cases}$$

Theorem 23 (Spectral Theorem - Commuting Operators Version)

Let A_1, \dots, A_n be a finite set of commuting, bounded, self-adjoint operators on a Hilbert space \mathcal{H} . There exist

- a measure space $\langle M, \mu \rangle$,
- bounded measurable functions $a_\ell : M \rightarrow \mathbb{R}$, $1 \leq \ell \leq n$, and
- a unitary operator $U : \mathcal{H} \rightarrow L^2(M, \mu)$

such that, for each $1 \leq \ell \leq n$,

$$(UA_\ell U^{-1}\varphi)(m) = a_\ell(m) \varphi(m)$$

for all $\varphi \in L^2(M, \mu)$. If \mathcal{H} is separable, μ can be chosen to be a finite measure.

Corollary 24 Let A be a bounded normal operator on a Hilbert space \mathcal{H} . There exist a measure space $\langle M, \mu \rangle$, a bounded measurable function $a : M \rightarrow \mathbb{C}$, and a unitary operator $U : \mathcal{H} \rightarrow L^2(M, \mu)$ such that

$$(UAU^{-1}\varphi)(m) = a(m) \varphi(m)$$

for all $\varphi \in L^2(M, \mu)$.

Theorem 25 (Spectral Theorem - Functional Calculus Version)

Let A be a bounded self-adjoint operator on a Hilbert space \mathcal{H} . Let $\mathcal{B} = \mathcal{B}([- \|A\|, \|A\|])$ denote the set of all bounded Borel functions on $[- \|A\|, \|A\|]$. There exists a unique map $\Phi : \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ such that

(a) Φ is an algebra $*$ -homomorphism. That is

$$\begin{aligned} \Phi(\alpha f + \beta g) &= \alpha \Phi(f) + \beta \Phi(g) & \Phi(\bar{f}) &= \Phi(f)^* \\ \Phi(fg) &= \Phi(f)\Phi(g) & \Phi(1) &= \mathbb{1} \end{aligned}$$

for all $f, g \in \mathcal{B}$ and $\alpha, \beta \in \mathbb{C}$.

(b) $\|\Phi(f)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_{L^\infty}$ for all $f \in \mathcal{B}$.

(c) $\Phi(x) = A$

(d) If the sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ converges pointwise to f and is uniformly bounded, then $\Phi(f) = \text{s-lim}_{n \rightarrow \infty} \Phi(f_n)$.

Furthermore, Φ has the following properties.

(e) If, for some $\lambda \in \mathbb{R}$ and $\psi \in \mathcal{H}$, we have $A\psi = \lambda\psi$, then $\Phi(f)\psi = f(\lambda)\psi$, for all $f \in \mathcal{B}$.

(f) If $f \in \mathcal{B}$ is nonnegative, then $\Phi(f)$ is a nonnegative operator.

(g) If $A, B \in \mathcal{L}(\mathcal{H})$ commute, then so do $\Phi(f)$ and B , for all $f \in \mathcal{B}$. That is, if $AB = BA$, then $\Phi(f)B = B\Phi(f)$.

Remark 26 It is common to write $f(A)$ in place of $\Phi(f)$. In this notation, the above conclusions are

(a)

$$\begin{aligned} (\alpha f + \beta g)(A) &= \alpha f(A) + \beta g(A) & \bar{f}(A) &= f(A)^* \\ (fg)(A) &= f(A)g(A) & 1(A) &= \mathbb{1} \end{aligned}$$

(b) $\|f(A)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_{L^\infty}$

(c) $x(A) = A$

(d) If $f_n \rightarrow f$ pointwise and is uniformly bounded, then $f(A) = \text{s-lim}_{n \rightarrow \infty} f_n(A)$.

(e) If $A\psi = \lambda\psi$, then $f(A)\psi = f(\lambda)\psi$.

(f) If $f \geq 0$, then $f(A) \geq 0$.

(g) If $AB = BA$, then $f(A)B = Bf(A)$.

Example 27 Let A be a self-adjoint, compact operator and let $\{\varphi_n\}_{n \in \mathcal{I}}$ be a complete orthonormal basis for \mathcal{H} consisting of eigenvectors of A . Denote by λ_n the eigenvalue of A for the eigenvector φ_n . We just have to define, for each $f \in \mathcal{B}(\mathbb{R})$

$$f(A) \left(\sum_{n \in \mathcal{I}} x_n \varphi_n \right) = \sum_{n \in \mathcal{I}} f(\lambda_n) x_n \varphi_n$$

Definition 28 (Projection Valued Measure) Denote by $\mathcal{B}_{\mathbb{R}}$ the σ -algebra of Borel subsets of \mathbb{R} and by $\mathcal{L}(\mathcal{H})$ the set of a bounded operators on \mathcal{H} . A projection valued measure is a map $E : \mathcal{B}_{\mathbb{R}} \rightarrow \mathcal{L}(\mathcal{H})$ that obeys the following conditions.

- (i) For each $B \in \mathcal{B}_{\mathbb{R}}$, the operator $E(B)$ is an orthogonal projection on some closed subspace of \mathcal{H} . That is, $E(B)^2 = E(B)$ and $E(B) = E(B)^*$.
- (ii) $E(\emptyset) = 0$ and $E(\mathbb{R}) = \mathbb{1}$
- (iii) If $\{B_n\}_{n \in \mathbb{N}}$ is a countable family of disjoint Borel subsets of \mathbb{R} , then

$$E\left(\bigcup_{n=1}^{\infty} B_n\right) = \text{s-lim}_{N \rightarrow \infty} \sum_{n=1}^N E(B_n)$$

A projection valued measure is said to be bounded if, in addition,

- (iv) There is an $a > 0$ such that $E((-a, a)) = \mathbb{1}$.

A projection valued measure automatically also obeys

- (v) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for all $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$.
- (vi) $E(B_1)$ and $E(B_2)$ commute for all $B_1, B_2 \in \mathcal{B}_{\mathbb{R}}$.

Definition 29 ($\int f(\lambda) dE(\lambda)$) Let $B \mapsto E(B)$ be a bounded projection valued measure, and $\varphi, \psi \in \mathcal{H}$. Observe that

- $B \mapsto \langle \varphi, E(B)\varphi \rangle$ is an ordinary finite Borel measure on \mathbb{R} .
- $B \mapsto \langle \varphi, E(B)\psi \rangle$ is an ordinary complex measure on \mathbb{R} . That is, there are (positive) measures μ_1, μ_2, ν_1 and ν_2 such that

$$\langle \varphi, E(B)\psi \rangle = \mu_1(B) - \mu_2(B) + i\nu_1(B) - i\nu_2(B)$$

By the polarization identity, we can take

$$\begin{aligned} \mu_1(B) &= \frac{1}{4} \langle \varphi + \psi, E(B)(\varphi + \psi) \rangle & \mu_2(B) &= \frac{1}{4} \langle \varphi - \psi, E(B)(\varphi - \psi) \rangle \\ \nu_1(B) &= \frac{1}{4} \langle \varphi + i\psi, E(B)(\varphi + i\psi) \rangle & \nu_2(B) &= \frac{1}{4} \langle \varphi - i\psi, E(B)(\varphi - i\psi) \rangle \end{aligned}$$

- Use $\mathcal{B}(\mathbb{R})$ to denote the set of bounded, Borel measurable functions on \mathbb{R} . If $f \in \mathcal{B}(\mathbb{R})$, then the map $(\varphi, \psi) \mapsto \int f(\lambda) d\langle \varphi, E(\lambda)\psi \rangle$ is well-defined, bounded and sesquilinear. So, by (a corollary to) the Riesz representation theorem, there is a unique $F \in \mathcal{L}(\mathcal{H})$ such that

$$\langle \varphi, F\psi \rangle = \int f(\lambda) d\langle \varphi, E(\lambda)\psi \rangle$$

for all $\varphi, \psi \in \mathcal{H}$.

We define

$$\int f(\lambda) dE(\lambda) = F$$

Example 30 Let $\mathcal{H} = L^2(M, \mu)$ for some measure space (M, μ) and let $a : M \rightarrow \mathbb{R}$ be bounded and measurable. Define, for each $B \in \mathcal{B}_{\mathbb{R}}$

$$E(B) = \text{multiplication by } \chi_{a^{-1}(B)}(m) = \chi_B(a(m))$$

This is a bounded projection valued measure and

$$\begin{aligned} \int f(\lambda) d\langle \varphi, E(\lambda)\varphi \rangle &= \int f(a(m)) |\varphi(m)|^2 d\mu(m) \\ \int f(\lambda) d\langle \varphi, E(\lambda)\psi \rangle &= \int f(a(m)) \overline{\varphi(m)}\psi(m) d\mu(m) \end{aligned}$$

Theorem 31 (Spectral Theorem - Projection-valued Measure Version)

There is a 1-1 correspondence between bounded self-adjoint operators and bounded projection-valued measures $A \leftrightarrow E_A$ such that

$$A = \int \lambda dE_A(\lambda)$$

Example 32 Let A be a self-adjoint, compact operator and let $\{\varphi_n\}_{n \in \mathcal{I}}$ be a complete orthonormal basis for \mathcal{H} consisting of eigenvectors of A . Denote by λ_n the eigenvalue of A for the eigenvector φ_n . Define, for each $m \in \mathcal{I}$, P_m to be the orthogonal projector onto the linear subspace of \mathcal{H} consisting of all scalar multiples of φ_m . That is

$$P_m \left(\sum_{n \in \mathcal{I}} x_n \varphi_n \right) = x_m \varphi_m$$

Set, for each Borel subset B of \mathbb{R}

$$E_A(B) = \sum_{\substack{m \in \mathcal{I} \\ \text{with } \lambda_m \in B}} P_m$$

Then

$$A = \sum_{m \in \mathcal{I}} \lambda_m P_m = \sum_{m \in \mathcal{I}} \lambda_m E_A(\{\lambda_m\}) = \int \lambda dE_A(\lambda)$$

Corollary 33

$$\begin{aligned} \lambda \in \sigma(A) &\iff E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0 \quad \text{for all } \varepsilon > 0 \\ \lambda \in \sigma_p(A) &\iff E_A(\{\lambda\}) \neq 0 \\ E_A(\rho(A) \cap \mathbb{R}) &= 0 \\ \text{range } E_A(\{0\}) &= \ker A \end{aligned}$$

Corollary 34 Let $-\infty < a < b < \infty$.

(a) Then

$$\frac{1}{2} \{E([a, b]) + E((a, b))\} = \text{s-lim}_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_a^b \{(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}\} d\lambda$$

(b) Let a and b be in the resolvent set of A . Then

$$E_A([a, b]) = E_A([a, b)) = E_A((a, b]) = E_A((a, b)) = \frac{1}{2\pi i} \int_{C_{a,b}} R_\zeta(A) d\zeta$$

for any simple closed curve $C_{a,b}$ in the complex plane with counterclockwise orientation that contains $\{\lambda + i0 \mid a \leq \lambda \leq b\}$ in its interior.

Definition 35 A Banach Algebra \mathcal{A} is a set which

(a) is a complex algebra, i.e. a set equipped with three operations

$$(A, B) \in \mathcal{A} \times \mathcal{A} \mapsto A+B \in \mathcal{A} \quad (\alpha, A) \in \mathbb{C} \times \mathcal{A} \mapsto \alpha A \in \mathcal{A} \quad (A, B) \in \mathcal{A} \times \mathcal{A} \mapsto AB \in \mathcal{A}$$

called addition, scalar multiplication and multiplication, that obey the usual vector space axioms and

$$\begin{aligned} A(BC) &= (AB)C & (A+B)C &= AB+BC & A(B+C) &= AB+AC \\ \alpha(BC) &= (\alpha B)C & &= B(\alpha C) & & \end{aligned}$$

for all $A, B, C \in \mathcal{A}$ and $\alpha \in \mathbb{C}$ (so multiplication is associative, but not necessarily commutative) and

(b) is normed, with the usual norm axioms, and also obeys $\|AB\| \leq \|A\| \|B\|$ for all $A, B \in \mathcal{A}$ and

(c) and is complete.

If, in addition,

(d) \mathcal{A} contains an identity element $\mathbb{1}$ that obeys $\|\mathbb{1}\| = 1$ and $A\mathbb{1} = \mathbb{1}A = A$ for all $A \in \mathcal{A}$ then \mathcal{A} is called a unital Banach algebra. Any Banach algebra can be easily extended to a unital Banach algebra $\mathcal{A} + \mathbb{1}$. (Define $\mathcal{A}_{\mathbb{1}} = \{(\alpha, A) \mid \alpha \in \mathbb{C}, A \in \mathcal{A}\}$, the algebraic operations by thinking of (α, A) as $\alpha\mathbb{1} + A$, and $\|(\alpha, A)\| = |\alpha| + \|A\|_{\mathcal{A}}$.)

Definition 36 A C^* -algebra is a Banach algebra \mathcal{A} together with a map $*$: $\mathcal{A} \rightarrow \mathcal{A}$ that obeys

$$\begin{aligned} (A+B)^* &= A^* + B^* & (\alpha A)^* &= \bar{\alpha} A^* & (AB)^* &= B^* A^* \\ A^{**} &= A & \|A^* A\| &= \|A\|^2 & & \end{aligned}$$

for all $A, B \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. (This used to be called a B^* -algebra.)

Remark 37 The condition $\|A^*A\| = \|A\|^2$ implies that $\|A^*\| = \|A\|$ since

$$\|A\|^2 = \|A^*A\| \leq \|A^*\| \|A\| \implies \|A\| \leq \|A^*\| \implies \|A^*\| \leq \|A^{**}\| = \|A\|$$

Example 38

(a) Let $A = A^* \in \mathcal{L}(\mathcal{H})$. Then $\mathcal{A} = \overline{\{\text{polynomials in } A \text{ with complex coefficients}\}}$, with the overbar denoting norm closure, is a commutative C^* -algebra contained in $\mathcal{L}(\mathcal{H})$.

(b) Let A_1, \dots, A_n be self-adjoint, commuting, bounded operators on \mathcal{H} . Then $\mathcal{A} = \overline{\{\text{polynomials in } A_1, \dots, A_n \text{ with complex coefficients}\}}$ is a commutative C^* -algebra in $\mathcal{L}(\mathcal{H})$.

(c) Any closed subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed under the taking of adjoints is a C^* -algebra. Conversely, any C^* -algebra is isomorphic to a subalgebra of $\mathcal{L}(\mathcal{H})$.

(d) Let X be a Hausdorff space (that is, a set equipped with open sets such that distinct points have disjoint open neighbourhoods). Then $\mathcal{A} = C(X)$, the set of all continuous functions on X , with the supremum norm, is a commutative C^* -algebra .

Theorem 39 (Gelfand–Naimark)

Let \mathcal{A} be a commutative C^ -algebra with identity. Then there is a compact Hausdorff space, X , (unique up to homeomorphism) such that \mathcal{A} is $*$ -isomorphic to $C(X)$. That is, there is a 1–1, onto map $\Psi : \mathcal{A} \rightarrow C(X)$ such that*

$$\begin{aligned} \Psi(A + B) &= \Psi(A) + \Psi(B) & \Psi(\alpha A) &= \alpha \Psi(A) & \Psi(AB) &= \Psi(A) \Psi(B) \\ \Psi(A^*) &= \overline{\Psi(A)} & \|\Psi(A)\|_{C(X)} &= \|A\|_{\mathcal{A}} \end{aligned}$$

for all $A, B \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.