MATH 511 Problem Set 6 Solutions

1. Denote by $AC[0,1]$ the set of all functions $f : [0,1] \to \mathbb{C}$ of the form $f(x) = C + \int_0^x \varphi(t) \, dt$ with $C \in \mathbb{C}$ and $\varphi \in L^2([0,1])$. For such a function, define $\frac{d}{dx}(x) = \varphi(x)$. Note that, by the Lebesgue dominated convergence theorem, any $f \in AC[0,1]$ is continuous. “AC” stands for “absolutely continuous”, but this is not the usual definition of absolutely continuous and in fact is not exactly equivalent to it. For the usual definition, see [Folland, Real Analysis, (3.31)] and for the relation between our definition and the usual definition, see [Folland, Real Analysis, Theorem 3.35].

Define $T : D(T) \subset L^2([0,1]) \to L^2([0,1])$ and $\tilde{T} : D(\tilde{T}) \subset L^2([0,1]) \to L^2([0,1])$ by

$$
D(T) = AC[0,1] \cap \{ \varphi : [0,1] \to \mathbb{C} \mid \varphi(0) = \varphi(1) = 0 \} \quad T\varphi = i \frac{d\varphi}{dx}
$$

$$
D(\tilde{T}) = AC[0,1] \quad \tilde{T}\varphi = i \frac{d\varphi}{dx}
$$

(a) Prove that if $f \in AC[0,1]$ and

$$
f(x) = C + \int_0^x \varphi(t) \, dt = D + \int_0^x \psi(t) \, dt
$$

with $C, D \in \mathbb{C}$ and $\varphi, \psi \in L^2[0,1]$, then $C = D$ and $\varphi = \psi$ a.e..

(b) Let $f : [0,1] \to \mathbb{C}$ be continuous on $[0,1]$ and have a bounded derivative, $g'$, on $(0,1)$. Prove that if $f \in AC[0,1]$, then $fg \in AC[0,1]$ and $\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$.

(c) Prove that $\tilde{T} \subset T^*$.

(d) Prove that if $\varphi \in D(T^*)$, then there is a subset $B \subset [0,1]$ of measure zero such that,

$$
\varphi(x) = \varphi(y) - i \int_y^x (T^*\varphi)(t) \, dt \quad \text{for all} \quad x, y \in [0,1] \setminus B
$$

Hint: If $0 < y < x < 1$, then $\int_y^x (T^*\varphi)(t) \, dt = \langle \chi_{[y,x]} , T^*\varphi \rangle$.

(e) Prove that $T^* = \tilde{T}$.

Solution. (a) Setting $x = 0$ gives

$$
f(0) = C = D
$$

So

$$
\int_0^x [\varphi(t) - \psi(t)] \, dt = 0
$$

for all $x \in [0,1]$. Thus $\varphi - \psi$ is orthogonal to $\chi_{[0,x]}$ for all $x \in [0,1]$. By taking linear combinations (and using that single point sets have measure zero) we have that $\varphi - \psi$ is orthogonal to all step functions. Since step functions are dense in $L^2([0,1])$, $\varphi - \psi = 0$ almost everywhere.

(b) Write $f(x) = f(0) + \int_0^x \varphi(t) \, dt$. Set $\Phi(t) = \varphi(t)g(t) + f(t)g'(t)$. It suffices to prove that

$$
f(x)g(x) = f(0)g(0) + \int_0^x \Phi(t) \, dt
$$

To do so, we compute

$$
\int_0^x \Phi(t) \, dt = \int_0^x \varphi(t)g(t) \, dt + \int_0^x f(s)g'(s) \, ds
$$

$$
= \int_0^x d\varphi(t) \left[ g(0) + \int_0^t g'(s) \right] + \int_0^x ds g'(s) \left[ f(0) + \int_0^s dt \varphi(t) \right]
$$

$$
= \int_0^x d\varphi(t) \left[ g(0) + \int_0^t ds g'(s) \right] + \int_0^x ds g'(s) \left[ f(0) + \int_0^s dt \varphi(t) \right]
$$

$$
= \int_0^x dt \varphi(t) \left[ \int_0^x ds g'(s) + [f(x) - f(0)]g(0) + f(0)[g(x) - g(0)] \right]
$$

$$
= \int_0^x dt \varphi(t) \left[ \int_0^x ds g'(s) + [f(x) - f(0)]g(0) + f(0)[g(x) - g(0)] \right]
$$

$$
= [f(x) - f(0)][g(x) - g(0)] + [f(x) - f(0)]g(0) + f(0)[g(x) - g(0)]
$$

$$
= f(x)g(x) - f(0)g(0)
$$
(c) Let \( \psi(x) = \psi(0) + \int_0^x \psi'(t) \, dt \in D(\tilde{T}) = AC[0,1] \) and \( \varphi(x) = \int_0^x \varphi'(t) \, dt \in D(T) \). We must verify that \( \langle \imath \psi', \varphi \rangle = \langle \psi, \imath \varphi' \rangle \).

\[
\langle \imath \psi', \varphi \rangle = -\imath \int_0^1 dx \, \overline{\psi'(x)} \left[ \int_0^x \varphi'(t) \, dt \right] \\
= \imath \int_0^1 dx \, \overline{\psi'(x)} \left[ \int_x^1 \varphi'(t) \, dt \right] \quad \text{since } \int_0^1 \varphi'(t) \, dt = 0 \\
= \imath \int_0^1 dx \int_0^1 dt \, \overline{\psi(x) \varphi'(t)} \\
= \imath \int_0^1 dt \, \varphi'(t) \left[ \int_0^1 \overline{\psi(x)} \, dx \right] \\
= \langle \psi, \imath \varphi' \rangle \quad \text{since } \int_0^1 \varphi'(t) \, dt = 0
\]

(d) Let \( \varphi \in D(T^*) \). So \( \langle \imath \psi', \varphi \rangle = \langle \psi, T^* \varphi \rangle \) for all \( \psi \in D(T) \).

We may, without loss of generality, restrict our attention to \( 0 < y < x < 1 \). As motivation for our derivation, observe that, if \( 0 < y < x < 1 \), then

\[
\int_y^x (T^* \varphi)(t) \, dt = \langle \chi_{[y,x]}, T^* \varphi \rangle
\]

If the characteristic function \( \chi_{[y,x]} \) were in the domain of \( T \), we would be able to compute

\[
\langle \chi_{[y,x]}, T^* \varphi \rangle = \langle T \chi_{[y,x]}, \varphi \rangle = \langle \imath \chi'_{[y,x]}, \varphi \rangle = -\imath \left[ \varphi(y) - \varphi(x) \right]
\]

since the derivative of the characteristic function \( \chi_{[y,x]} \) is a delta function at \( y \) minus a delta function at \( x \). Of course this computation was not legitimate because \( \chi_{[y,x]} \) is not in the domain of \( T \). To legitimize it, we express \( \chi_{y,x} \) as a limit of \( C_0^\infty \) functions.

Fix any nonnegative \( C_0^\infty \) function \( \delta_1(t) \) that is supported on \([-1,1]\) and obeys \( \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \). Set for each \( \varepsilon > 0 \), \( \delta_{\varepsilon}(t) = \frac{1}{2\varepsilon} \delta\left( \frac{t}{\varepsilon} \right) \). Observe that \( \delta_{\varepsilon}(t) \) is nonnegative and supported on \([-\varepsilon,\varepsilon]\) and obeys \( \int_{-\infty}^{\infty} \delta_{\varepsilon}(t) \, dx = 1 \). Define, for each \( 0 < \varepsilon < \frac{1}{4} \) and \( \varepsilon < y < x < 1 - \varepsilon \) with \( x \geq y + 2\varepsilon \),

\[
\chi_{\varepsilon,y,x}(t) = \int_{-\infty}^t \left[ \delta_{\varepsilon}(s - y) - \delta_{\varepsilon}(s - x) \right] \, ds
\]

Here is a sketch of the graph of the integrand \( \delta_{\varepsilon}(s - y) - \delta_{\varepsilon}(s - x) \).
We have \( \chi_{\varepsilon, y, x} \in C_0^\infty(\mathbb{R}) \) and obeys \( 0 \leq \chi_{\varepsilon, y, x} \leq 1 \) and
\[
\chi_{\varepsilon, y, x}(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq y - \varepsilon \\
1 & \text{if } y + \varepsilon \leq t \leq x - \varepsilon \\
0 & \text{if } x + \varepsilon \leq t \leq 1 
\end{cases} \]
\( \chi_{\varepsilon, y, x}'(t) = \delta_x(t - y) - \delta_x(t - x) \)

We have \( \chi_{\varepsilon, y, x}(t) \in D(T) \), so that, for any \( \varphi \in D(T^*) \),
\[
\int_0^1 \chi_{\varepsilon, y, x}(t) (T^* \varphi)(t) \, dt = \langle \chi_{\varepsilon, y, x}(t), T^* \varphi \rangle = \langle T \chi_{\varepsilon, y, x}(t), \varphi \rangle = -i \int_0^1 \chi_{\varepsilon, y, x}'(t) \varphi(t) \, dt \\
= -i \int_{y+\varepsilon}^{y+\varepsilon} \delta_x(t - y) \varphi(t) \, dt + i \int_{x-\varepsilon}^{x-\varepsilon} \delta_x(t - x) \varphi(t) \, dt
\]
When we take the limit \( \varepsilon \to 0 \) the left hand side converges to \( \int_y^x (T^* \varphi)(t) \, dt \) for all \( 0 < y < x < 1 \) by the Lebesgue dominated convergence theorem. As far as the right hand side is concerned,
\( \circ \) if \( \varphi \) is continuous, then \( \lim_{\varepsilon \to 0} \int_0^1 \chi_{\varepsilon, y, x}'(t) \varphi(t) \, dt = \varphi(y) - \varphi(x) \), for each \( 0 < y < x < 1 \).
\( \circ \) The map
\[
\varphi(x) \in L^2([0,1]) \mapsto (G_{\varepsilon} \varphi)(x) = \int_{\min\{x+\varepsilon,1\}}^{\max\{x-\varepsilon,0\}} \delta_x(t - x) \varphi(t) \, dt \in L^2([0,1])
\]
is a bounded linear map of norm at most one by Problem Set 1, \#8a.

\( \circ \) Consequently, by an \( \frac{\pi}{2} \) argument, \((G_{\varepsilon} \varphi)(x)\) converges in \( L^2 \) to \( \varphi(x) \) as \( \varepsilon \to 0 \). So there is a subsequence of \( \varepsilon \)'s such that \( (G_{\varepsilon} \varphi)(x) \) converges pointwise, except on a set \( B \) of measure zero, to \( \varphi(x) \) as \( \varepsilon \to 0 \).

\( \circ \) Consequently there is a subsequence of \( \varepsilon \)'s such that \( -i \int_0^1 \chi_{\varepsilon, y, x}'(t) \varphi(t) \, dt \) converges for each \( x \) and \( y \) not in \( B \) to \( -i \varphi(y) + i \varphi(x) \) as \( \varepsilon \to 0 \).

(e) We already know from part (c) that \( \hat{T} \subset T^* \), so it suffices to prove that \( T^* \subset \hat{T} \). Let \( \varphi \in D(T^*) \).

Part (d) implies that \( \varphi \in AC[0,1] \) and \( \varphi' = iT^* \varphi \). To see that this is the case, fix any \( y \in [0,1] \setminus B \) and set \( C = \varphi(y) + i \int_0^y (T^* \varphi)(t) \, dt \) and \( \Phi = -i T^* \varphi \). Then \( \varphi(x) = C + \int_0^x \Phi(t) \, dt \) a.e.

2. This is a continuation of Problem Set 5, \# 6. Let \( \mathcal{H} \) be a Hilbert space and \( A : D(A) \subset \mathcal{H} \to \mathcal{H} \) be a closed linear operator. Let \( 0 \leq r < 1, R \in [0, \infty) \) and let \( B : D(B) \subset \mathcal{H} \to \mathcal{H} \) be another linear operator with \( D(A) \subset D(B) \) and
\[
\|B \varphi\| \leq r \|A \varphi\| + R \|\varphi\| \quad \text{for all } \varphi \in D(A)
\]

(c) Assume that \( A \) is self–adjoint and \( B \) is symmetric. Let \( \hat{D} \) be a linear subspace of \( D(A) \). Prove that if \( A \) is essentially self–adjoint on \( \hat{D} \), then \( A + B \) is again essentially self–adjoint on \( \hat{D} \).

Solution. By part (b), \( A + B \) (with domain \( D(A) \)) is self–adjoint. So it suffices to prove that
\[
\overline{A + B} \cap \mathcal{H} = A + B \quad \text{(with domain } D(A))
\]

We do so. Let \( \{\varphi_n\}_{n \in \mathbb{N}} \subset \hat{D} \) obey
\[
\lim_{n \to \infty} \varphi_n = \varphi \quad \lim_{n \to \infty} (A+B)\varphi_n = \psi
\]
By part (a), \( A + B \), with domain \( D(A) \), is closed so that \( \varphi \in D(A) \) and \( (A+B)\varphi = \psi \). Hence \( \overline{A+B} \cap \mathcal{H} \subset A+B \) (with domain \( D(A) \)). Conversely, let \( \varphi \in D(A) \). Since \( A \) is essentially self–adjoint on \( \hat{D} \), there is a sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \subset \hat{D} \) with \( \lim_{n \to \infty} \varphi_n = \varphi \) and \( \lim_{n \to \infty} A \varphi_n = A \varphi \). Since
\[
\|B \varphi - B \varphi_n\| \leq r \|A \varphi - A \varphi_n\| + R \|\varphi - \varphi_n\| \xrightarrow{n \to \infty} 0
\]
we have \( \lim_{n \to \infty} (A+B) \varphi_n = (A+B) \varphi \) and \( A+B \subset \overline{A+B} \cap \mathcal{H} \).
3. Let \( \mathcal{H} = \ell^2 \).

(a) Find an isometric, linear operator whose domain is \( \ell^2 \) and whose range is a proper subset of \( \ell^2 \).
(b) Find a symmetric operator that has no self–adjoint extensions.

**Solution.** (a) The right shift operator \( R : \ell^2 \to \ell^2 \), defined by

\[
R(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, \alpha_1, \alpha_2, \alpha_3, \cdots)
\]

does the job. It obeys \( \|R\alpha\| = \|\alpha\| \), has domain \( \ell^2 \) and range \( \{ (\beta_1, \beta_2, \beta_3, \cdots) \in \ell^2 \mid \beta_1 = 0 \} \).

(b) Define \( A = C_-(R) = i(1 + R)(1 - R)^{-1} \), the inverse Cayley transform of the right shift operator, \( R \), of part (a). Observe that the point spectrum of \( R \) is the empty set and, in particular, that 1 is not an eigenvalue of \( R \). So \( A \) is self–adjoint.

\( \dim(\mathcal{H}) = \dim(\sigma(A)) = 1 \).

\( C_+(A) = R \):

Observe that, on the domain of \( A \), i.e. on the range of \( 1 - R \),

\[
A + iI = i(1 + R)(1 - R)^{-1} + i(1 - R)(1 - R)^{-1} = 2i(1 - R)^{-1}
\]
\[
A - iI = i(1 + R)(1 - R)^{-1} - i(1 - R)(1 - R)^{-1} = 2iR(1 - R)^{-1}
\]

This tells us that \((A + iI)\) has an inverse and \((A + iI)^{-1} = \frac{1}{2i}(1 - R)\). In particular the domain of \((A + iI)^{-1}\) is \( \ell^2 \). Furthermore

\[
C_+(A) = (A - iI)(A + iI)^{-1} = 2iR(1 - R)^{-1} \frac{1}{2i}(1 - R) = R
\]

\( \dim D(C_+(A)) = \dim D(R) = \dim (\ell^2) = 0 \)

\( \dim R(C_+(A)) = \dim R(R) = 1 \). (In \( R(R) \), the left \( R \) stands for “range” and the right \( R \) is the right shift operator.) So, by an in–class theorem, \( A \) has no self adjoint extensions.

**Remark:** We can compute \( A \) explicitly and do so now, starting with a determination of \((1 - R)^{-1}\). As

\[
R(\alpha_1, \alpha_2, \alpha_3, \cdots) = (0, \alpha_1, \alpha_2, \alpha_3, \cdots)
\]
\[
(1 - R)(\alpha_1, \alpha_2, \alpha_3, \cdots) = (\alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \cdots)
\]

we have \((1 - R)\tilde{\alpha} = \tilde{\beta}\) if and only if

\[
\begin{align*}
\beta_1 &= \alpha_1 \\
\beta_2 &= \alpha_2 - \alpha_1 \\
\beta_3 &= \alpha_3 - \alpha_2 \\
\end{align*}
\]

\[
\begin{align*}
\alpha_1 &= \beta_1 \\
\alpha_2 &= \beta_2 + \alpha_1 = \beta_1 + \beta_2 \\
\alpha_3 &= \beta_3 + \alpha_2 = \beta_1 + \beta_2 + \beta_3 \\
\end{align*}
\]
Thus

\[
([\| - R]^{-1}\tilde{\beta})_k = \sum_{j=1}^k \beta_j
\]

\[
(R[\| - R]^{-1}\tilde{\beta})_k = \sum_{j=1}^{k-1} \beta_j
\]

\[
(A\tilde{\beta})_k = (C_-(R)\tilde{\beta})_k = (i[\| + R][\| - R]^{-1}\tilde{\beta})_k = i\left(\beta_k + 2 \sum_{j=1}^{k-1} \beta_j\right)
\]

with

\[
D(A) = D([\| - R]^{-1}) = \text{range} (\| - R) = \left\{ \tilde{\beta} \in \ell^2 \mid \tilde{\alpha} = \left(\sum_{j=1}^k \beta_j\right)_{k \in \mathbb{N}} \in \ell^2 \right\} = \left\{ (\alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \cdots) \mid \tilde{\alpha} \in \ell^2 \right\}
\]

**Remark.** Let \( T \) be a symmetric operator. Recall the strategy that we use to construct self-adjoint extensions of \( T \). First we construct the Cayley transform \( C_+(T) = (T - i\|)(T + i\|)^{-1} \) of \( T \). We know that

- \( C_+(T) \) is a well-defined, isometric operator with domain \( R(T + i\|) \) and range \( \overline{R(T - i\|)} \).
- another symmetric operator \( S \) is an extension of \( T \) if and only if \( C_+(T) \subset C_+(S) \). (Recall that \( T = C_-(C_+(T)) \).
- The map \( S \mapsto C_+(S) \) provides a one–to–one correspondance between

\[
\{ S : D(S) \subset \mathcal{H} \rightarrow \mathcal{H} \mid S \text{ self–adjoint} \} \text{ and } \{ U : \mathcal{H} \rightarrow \mathcal{H} \mid U \text{ unitary}, 1 \notin \sigma_p(U) \}
\]

So there is a one–to–one correspondance between self–adjoint extensions of \( T \) and unitary operators \( U \) that

(a) extend the isometric operator \( C_+(T) \) and
(b) obey \( 1 \notin \sigma_p(U) \).

First forget about the condition that \( 1 \notin \sigma_p(U) \). It’s a simple matter to construct all unitary operators that extend \( C_+(T) \). Since the closure \( \overline{C_+(T)} \) is an isometric operator with domain \( R(T + i\|) \) and range \( \overline{R(T - i\|)} \), there is a one–to–one–correspondance between such unitary extensions and unitary maps \( \hat{U} : R(T + i\|)^\perp \rightarrow R(T - i\|)^\perp \). Now back to condition (b). The following problem shows it is automatically satisfied!

4. Let \( \mathcal{H} \) be a Hilbert space, \( D(T) \) a dense linear subspace of \( \mathcal{H} \) and \( T : D(T) \rightarrow \mathcal{H} \) a symmetric operator. Suppose that \( U \) is a unitary extension of \( C_+(T) \). Prove that \( 1 \notin \sigma_p(U) \).

**Solution.** Let \( U\varphi = \varphi \). We must show that \( \varphi = 0 \). Since \( \varphi \in \ker (\| - U) \), we have \( \varphi \in R(\| - U^*)^\perp \).

Since \( U \) is unitary the range of \( \| - U^* = (U - \|)U^* \) is the same as the range of \( \| - U \). So \( \varphi \in R(\| - U)^\perp \).

Since \( U \) is an extension of \( C_+(T) \), \( \varphi \in R(\| - C_+(T))^\perp \). But, on the domain of \( C_+(T) \),

\[
\| - C_+(T) = (T + i\|)(T + i\|)^{-1} - (T - i\|)(T + i\|)^{-1} = 2i(T + i\|)^{-1}
\]

so that the range of \( R(\| - C_+(T)) = D(T) \) and \( \varphi \in D(T)^\perp \). Since \( D(T) \) is dense, \( \varphi = 0 \).