1. Let \( \{ e_n \}_{n \in \mathbb{N}} \) be an orthonormal basis for a Hilbert space \( \mathcal{H} \) and set \( e_\infty = \sum_{j=1}^\infty 2^{-j} e_j \). Define the linear operator \( T : D(T) \subset \mathcal{H} \rightarrow \Phi \) by

\[
D(T) = \left\{ \alpha e_\infty + \sum_{j=1}^N \beta_j e_j \mid N \in \mathbb{N}, \alpha, \beta_1, \ldots, \beta_N \in \Phi \right\}
\]

Then

\[
T(\alpha e_\infty + \sum_{j=1}^N \beta_j e_j) = \alpha
\]

Prove that \( T \) is a well-defined linear operator but is not closable.

**Solution.** well-defined: If

\[
\alpha e_\infty + \sum_{j=1}^N \beta_j e_j = \alpha' e_\infty + \sum_{j=1}^{N'} \beta'_j e_j
\]

then

\[
\sum_{j=1}^\infty \gamma_j e_j = 0 \quad \text{with} \quad \gamma_j = \begin{cases} 2^{-j}(\alpha - \alpha') & \text{if } j > \max\{N, N'\} \\ 2^{-j}(\alpha - \alpha') + \beta_j & \text{if } N' < j \leq N \\ 2^{-j}(\alpha - \alpha') - \beta'_j & \text{if } N < j \leq N' \\ 2^{-j}(\alpha - \alpha') + (\beta_j - \beta'_j) & \text{if } j \leq \min\{N, N'\} \end{cases}
\]

Using some \( j > \max\{N, N'\} \) gives \( \alpha = \alpha' \) and hence

\[
T(\alpha e_\infty + \sum_{j=1}^N \beta_j e_j) = \alpha = \alpha' = T(\alpha' e_\infty + \sum_{j=1}^{N'} \beta'_j e_j)
\]

**linearity:** is obvious.

**not closable:** Let, for each \( m \in \mathbb{N} \), \( \varphi_m = \sum_{j=1}^m 2^{-j} e_j \). This is a sequence of vectors in \( D(T) \) with

\[
\lim_{m \rightarrow \infty} \varphi_m = e_\infty \quad \lim_{m \rightarrow \infty} T\varphi_m = 0 \neq 1 = T\left( \lim_{m \rightarrow \infty} \varphi_m \right)
\]

2. Give four examples of closed multiplication operators with

(a) one example having both domain and range closed and
(b) one example having domain closed but range not closed and
(c) one example having range closed but domain not closed and
(d) one example having both domain and range not closed.

**Solution.** All of our examples will be multiplication operators \( A : \varphi(x) \mapsto a(x)\varphi(x) \) on dense subsets of \( L^2(\mathbb{R}) \). We’ll use \( D(A) \) and \( R(A) \) to denote the domain and range of \( A \), respectively. Recall that \( C_0^\infty(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \).

(a) \( a(x) = 1 \), \( D(A) = L^2(\mathbb{R}) \), \( R(A) = L^2(\mathbb{R}) \). A is closed because it is bounded and has closed domain.

(b) \( a(x) = \frac{1}{1+x^2} \), \( D(A) = L^2(\mathbb{R}) \), \( C_0^\infty(\mathbb{R}) \subset R(A) = \left\{ \frac{\psi(x)}{1+x^2} \mid \psi \in L^2(\mathbb{R}) \right\} \), \( \frac{1}{1+x^2} \notin R(A) \). A is closed because it is bounded and has closed domain.

(c) \( a(x) = 1 + x^2 \), \( C_0^\infty(\mathbb{R}) \subset D(A) = \left\{ \psi \in L^2(\mathbb{R}) \mid (1+x^2)\psi(x) \in L^2(\mathbb{R}) \right\} \), \( R(A) = L^2(\mathbb{R}) \), \( \frac{1}{1+x^2} \notin D(A) \). A is closed because \( A^{-1} \) is bounded and has closed domain.

(d) \( a(x) = e^x \), \( D(A) = \left\{ \varphi \in L^2(\mathbb{R}) \mid e^x\varphi(x) \in L^2(\mathbb{R}) \right\} \), \( R(A) = \left\{ \psi \in L^2(\mathbb{R}) \mid e^{-x}\psi(x) \in L^2(\mathbb{R}) \right\} \).

Then \( C_0^\infty(\mathbb{R}) \subset D(A) \cap R(A) \), but \( \frac{1}{e^x} \notin D(A) \cup R(A) \). To see that \( A \) is closed, assume that \( \{ \varphi_n \}_{n \in \mathbb{N}} \) is a sequence in \( D(A) \) that converges in \( L^2(\mathbb{R}) \) to \( \Phi \) and assume further that \( \{ e^x\varphi_n \}_{n \in \mathbb{N}} \) converges in \( L^2(\mathbb{R}) \) to \( \Psi \). Multiplication by both \( \chi_{[0,\infty)} \) and \( \chi_{[0,\infty)} e^{-x} \) are bounded operators, so that \( \{ \chi_{[0,\infty)} e^x \varphi_n \}_{n \in \mathbb{N}} \) converges to both \( \chi_{[0,\infty)} \Phi \) and \( \chi_{[0,\infty)} e^{-x} \Psi \). Thus \( \chi_{[0,\infty)} \Phi = \chi_{[0,\infty)} e^{-x} \Psi \). Similarly, multiplication by both \( \chi_{(-\infty,0]} \) and \( \chi_{(-\infty,0]} e^x \) are bounded operators, so that \( \{ \chi_{(-\infty,0]} e^x \varphi_n \}_{n \in \mathbb{N}} \) converges to both \( \chi_{(-\infty,0]} e^x \Phi \) and \( \chi_{(-\infty,0]} \Psi \). Thus \( \chi_{(-\infty,0]} e^x \Phi = \chi_{(-\infty,0]} \Psi \). All together \( e^x \Phi = \Psi \) so that \( \Phi \in D(A) \) and \( A\Phi = \Psi \).
3. Let $\mathcal{H}$ be a Hilbert space and $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator. Prove that if $\varphi \perp \text{range}(T)$, then $\varphi \in D(T^*)$ and $T^*\varphi = 0$.

**Solution.** If $\varphi$ is perpendicular to the range of $T$, then, for all $\psi \in D(T)$, $T\psi \in \text{range}(T)$ and

$$0 = \langle T\psi, \varphi \rangle = \langle \psi, 0 \rangle$$

Hence, by definition, $\varphi \in D(T^*)$ and $T^*\varphi = 0$.

4. Let $\mathcal{H}$ be a Hilbert space and $D(A)$ be a dense linear subspace of $\mathcal{H}$. Let $A : D(A) \to \mathcal{H}$ be a symmetric linear operator. Let $0 \neq \mu_0 \in \mathbb{R}$. Prove that if the range $R(A + i\mu_0\mathbb{I}) = \mathcal{H}$, then $R(A + i\mu\mathbb{I}) = \mathcal{H}$ for all nonzero $\mu \in \mathbb{R}$ of the same sign as $\mu_0$.

**Solution.** It suffices to consider the case $\mu_0 > 0$ since if $\mu_0 < 0$ and $R(A + i\mu_0\mathbb{I}) = \mathcal{H}$, then $-A$ is symmetric and $R(-A + (-\mu_0)\mathbb{I}) = \mathcal{H}$.

So fix any $\mu_0 > 0$. We first consider $\mu \in (0, 2\mu_0)$. Observe that if $\varphi \in D(A)$, then

$$\| (A + i\mu_0\mathbb{I})\varphi \|^2 = \langle (A + i\mu_0\mathbb{I})\varphi, (A + i\mu_0\mathbb{I})\varphi \rangle$$

$$= \text{Re} \left\{ \langle A\varphi, A\varphi \rangle + i\mu_0 \langle A\varphi, \varphi \rangle - i\mu_0 \langle \varphi, A\varphi \rangle + \mu_0^2 \langle \varphi, \varphi \rangle \right\}$$

$$= \| A\varphi \|^2 + \mu_0^2 \| \varphi \|^2 \geq \mu_0^2 \| \varphi \|^2$$

This implies that the operator $A + i\mu_0\mathbb{I}$ is injective, surjective (by hypothesis) and has a bounded inverse $(A + i\mu_0\mathbb{I})^{-1}$ with domain $\mathcal{H}$ and range $D(A)$ and operator norm at most $\frac{1}{\mu_0}$. Consequently, on $D(A)$,

$$A + i\mu\mathbb{I} = (A + i\mu_0\mathbb{I}) + i(\mu - \mu_0)\mathbb{I} = [\mathbb{I} + C](A + i\mu_0\mathbb{I})$$

with $C = i(\mu - \mu_0)(A + i\mu_0\mathbb{I})^{-1}$.

Since $\|C\| \leq |\mu - \mu_0| \|(A + i\mu_0\mathbb{I})^{-1}\| \leq |\mu - \mu_0| \frac{1}{\mu_0} < 1$, the operator $\mathbb{I} + C$ is bijective with a bounded inverse. As $A + i\mu_0\mathbb{I}$ is surjective, so is $\mathbb{I} + C(A + i\mu_0\mathbb{I}) = A + i\mu\mathbb{I}$. Thus $R(A + i\mu\mathbb{I}) = \mathcal{H}$ for all $\mu \in (0, 2\mu_0)$.

Replacing $\mu_0$ by $\frac{2}{3}\mu_0 \in (0, 2\mu_0)$, we have that $R(A + i\mu\mathbb{I}) = \mathcal{H}$ for all $\mu \in (0, 2 \times \frac{3}{2}\mu_0 = 3\mu_0)$. Replacing $\mu_0$ by $2\mu_0 \in (0, 3\mu_0)$, we have that $R(A + i\mu\mathbb{I}) = \mathcal{H}$ for all $\mu \in (0, 2 \times 2\mu_0 = 4\mu_0)$. Replacing $\mu_0$ by $3\mu_0 \in (0, 4\mu_0)$, we have that $R(A + i\mu\mathbb{I}) = \mathcal{H}$ for all $\mu \in (0, 2 \times 3\mu_0 = 6\mu_0)$. And so on.

**Remark:** The above argument requires requires $|\mu - \mu_0| < |\mu_0|$, but does not require $\mu$ to be real. So it shows that $R(A + i\mu\mathbb{I}) = \mathcal{H}$ for all $\mu$ in

$$\left\{ \mu \in \mathbb{C} \mid |\mu - \lambda\mu_0| < \lambda|\mu_0| \text{ for some } \lambda > 0 \right\} = \left\{ \mu \in \mathbb{C} \mid \text{Re } \mu \neq 0 \text{ and Re } \mu \text{ has the same sign as } \mu_0 \right\}$$
5. Let $\mathcal{H} = L^2(\mathbb{R})$,

$$D(A) = \left\{ f \in L^2(\mathbb{R}) \mid (1 + k^2)f(k) \in L^2(\mathbb{R}), \int f(k) \, dk = 0, \int kf(k) \, dk = 0 \right\}$$

and $A : D(A) \to L^2(\mathbb{R})$ be defined by

$$(Af)(k) = (1 + k^2)f(k)$$

Find $A^*$.

**Solution.** We showed in class that $D(A)$ is dense. So $A^*$ is well–defined. Fix any $\psi, \psi_* \in L^2(\mathbb{R})$. By definition $\psi \in D(A^*)$ with $\psi_* = A^*\psi$ if and only if

$$\langle \varphi, \psi_* \rangle = (A\varphi, \psi) = \int (1 + k^2)\overline{\varphi(k)}\psi(k) \, dk \iff \int \overline{\varphi(k)} [\psi_* - (1 + k^2)\psi] \, dk = 0$$

for all $\varphi \in D(A)$. Set $g(k) = \psi_*(k) - (1 + k^2)\psi(k)$. Note that $g(k)$ is a measurable function but need not be in $L^2(\mathbb{R})$. On the other hand $\frac{g(k)}{1 + k^2} \in L^2(\mathbb{R})$ and, if $\varphi \in D(A)$, then $(1 + k^2)\varphi \in L^2(\mathbb{R})$ so that then $\overline{\varphi(k)}g(k) \in L^1(\mathbb{R})$. Assume that $\int \overline{\varphi(k)}g(k) \, dk = 0$ for all $\varphi \in D(A)$. Then, for all $\varphi \in D(A)$,

$$0 = \int \overline{\varphi(k)}g(k) \, dk = \int (1 + k^2)\overline{\varphi(k)} \frac{g(k)}{1 + k^2} \, dk$$

This is the case if and only if $\frac{g(k)}{1 + k^2}$ is in the orthogonal complement of $(1 + k^2)D(A) = R(A)$ or

$$\frac{g(k)}{1 + k^2} \in \text{span}\{p_0, p_1\} \quad \text{where } p_0(k) = \frac{1}{1 + k^2}, \quad p_1(k) = \frac{k}{1 + k^2}$$

or $\psi_*(k) - \psi(k) = -p(k)$ with $p \in \text{span}\{p_0, p_1\}$

or $\psi_*(k) = (1 + k^2)(\psi(k) - p(k))$ with $p \in \text{span}\{p_0, p_1\}$

Recalling that $\psi, \psi_* \in L^2(\mathbb{R})$, this is the case if and only if (writing $\psi(k) - p(k) = \varphi(k)$)

$$\psi(k) = \varphi(k) + p(k), \quad \psi_*(k) = (1 + k^2)\varphi(k) \quad \text{with } \varphi(k) \in L^2(\mathbb{R}), \quad (1 + k^2)\varphi(k) \in L^2(\mathbb{R}), \quad p \in \text{span}\{p_0, p_1\}$$

In conclusion

$$D(A^*) = \left\{ \varphi(k) + p(k) \mid (1 + k^2)\varphi(k) \in L^2(\mathbb{R}), \quad p \in \text{span}\{p_0, p_1\} \right\} \quad (A^*(\varphi + p))(k) = (1 + k^2)\varphi(k)$$

6. Let $\mathcal{H}$ be a Hilbert space and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a closed linear operator. Let $0 \leq r < 1, \ R \in [0, \infty)$ and let $B : D(B) \subset \mathcal{H} \to \mathcal{H}$ be another linear operator with $D(A) \subset D(B)$ and

$$\|B\varphi\| \leq r\|A\varphi\| + R\|\varphi\| \quad \text{for all } \varphi \in D(A)$$

($B$ is called a Kato perturbation of $A.$)

(a) Prove that $A + B$, with domain $D(A + B) = D(A)$, is again a closed operator.

(b) Prove that if $A$ is self–adjoint and $B$ is symmetric then, $A + B$ with domain $D(A + B) = D(A)$, is again self–adjoint.

**Hint:** Show that if $\mu > 0$ is large enough, then $A + B \pm i\mu \mathbb{I} = \{\mathbb{I} + B(A \pm i\mu \mathbb{I})^{-1}\}(A \pm i\mu \mathbb{I})$ has range $\mathcal{H}$.
Solution. (a) Let \( \{ \varphi_n \}_{n \in \mathbb{N}} \subset D(A + B) = D(A) \) be sequence such that both \( \lim_{n \to \infty} \varphi_n = \varphi \) and \( \lim_{n \to \infty} (A + B)\varphi_n = \psi \). Then

\[
\|A(\varphi_n - \varphi_m)\| \leq \|(A + B)(\varphi_n - \varphi_m)\| + \|B(\varphi_n - \varphi_m)\| \\
\leq \|(A + B)(\varphi_n - \varphi_m)\| + \|A(\varphi_n - \varphi_m)\| + R\|\varphi_n - \varphi_m\|
\]

implies

\[
\|A(\varphi_n - \varphi_m)\| \leq \frac{1}{1 - R}\|(A + B)(\varphi_n - \varphi_m)\| + \frac{R}{1 - R}\|\varphi_n - \varphi_m\|
\]

Consequently, the sequence \( \{ A\varphi_n \}_{n \in \mathbb{N}} \) is a Cauchy sequence and hence converges. As \( A \) is closed, \( B \in D(A) \) and \( \lim_{n \to \infty} A\varphi_n = A\varphi \). As

\[
\|B\varphi - B\varphi_n\| \leq \|A\varphi - A\varphi_n\| + R\|\varphi - \varphi_n\|
\]

we also have \( \lim_{n \to \infty} B\varphi_n = B\varphi \) and

\[
\psi = \lim_{n \to \infty} (A + B)\varphi_n = (A + B)\varphi
\]

So \( A + B \) is closed.

(b) As \( A + B \) is clearly symmetric it suffices to prove that \( R(A + B \pm i\|) = \mathcal{H} \). So, by problem 4, it suffices to prove that \( R(A + B \pm i\mu\|) = \mathcal{H} \) for some \( \mu > 0 \). (I will later choose \( \mu \) large.) Since \( A \) is self-adjoint, \( A \pm i\mu\| \) has an everywhere defined, bounded inverse. Furthermore, for all \( \varphi \in D(A) \)

\[
\|(A \pm i\mu\|)\varphi\|^2 = \|A\varphi\|^2 + \mu^2\|\varphi\|^2
\]

Setting \( \varphi = (A \pm i\mu\|)^{-1}\psi \), we have \( \|A(A \pm i\mu\|)^{-1}\psi\|^2 + \mu^2\|(A \pm i\mu\|)^{-1}\psi\|^2 = \|\psi\|^2 \) and hence

\[
\|A(A \pm i\mu\|)^{-1}\| \leq 1 \quad \|(A \pm i\mu\|)^{-1}\| \leq \frac{1}{\mu}
\]

Consequently, for all \( \psi \in \mathcal{H} \)

\[
\|B(A \pm i\mu\|)^{-1}\psi\| \leq \|A(A \pm i\mu\|)^{-1}\psi\| + R\|(A \pm i\mu\|)^{-1}\psi\| \leq (r + \frac{R}{\mu})\|\psi\|
\]

Now pick \( \mu \) large enough that \( r + \frac{R}{\mu} < 1 \). By Problem Set 2 #7, the operator \( \| + B(A \pm i\mu\|)^{-1} \) has an everywhere defined, bounded inverse, and in particular has range \( \mathcal{H} \). Since \( A \) is self-adjoint, \( A \pm i\mu\| \) also has range \( \mathcal{H} \) and so

\[
A + B \pm i\mu\| = \left( \| + B(A \pm i\mu\|)^{-1} \right)(A \pm i\mu\|)
\]

also has range \( \mathcal{H} \).

7. Let \( D(T) \) be a dense subset of a Hilbert space \( \mathcal{H} \) and let \( T : D(T) \to \mathcal{H} \) be a closed linear operator. Two of the following statements are true. Prove them. One of the following statements is not true for at least one closed unbounded operator, though it is true for bounded operators. Exhibit a counterexample and say where the proof for the bounded case fails for the unbounded case.

(a) \( \rho(T) \) is an open subset of \( \mathbb{C} \)

(b) \( R_\lambda(T) = (\lambda\| - T)^{-1} \) is an analytic function of \( \lambda \) on \( \rho(T) \).

(c) \( \sigma(T) \neq \emptyset \)

Solution. (a) is true. To prove it, we extend the result of Problem Set 2, #7 and apply it with \( B = \lambda\| - T \) and \( C = \varepsilon\| \) with \( \lambda \in \rho(T) \) and \( \varepsilon \in \mathbb{C} \) obeying \( |\varepsilon| < \|\lambda\|^{-1} \).
**Lemma.** Let $D(B)$ be a dense subset of a Hilbert space $\mathcal{H}$ and let $B : D(B) \to \mathcal{H}$ be a linear operator that is 1–1 and onto and has a bounded inverse. Assume that $C : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator with $\|C\| < \|B^{-1}\|^{-1}$. Then $B + C$ is a bijection from $D(B)$ onto $\mathcal{H}$ and has a bounded inverse and furthermore

$$\|(B + C)^{-1} - B^{-1}\| \leq \frac{\|B^{-1}\|^2 \|C\|}{1 - \|B^{-1}\| \|C\|}$$

**Proof:** Write $r = \|C\| \|B^{-1}\| < 1$ and observe that $\|(CB^{-1})^n\| \leq r^n$. This implies that

$$D = \lim_{N \to \infty} \sum_{n=0}^{N} B^{-1}(-CB^{-1})^n \quad \text{and} \quad A = \lim_{N \to \infty} \sum_{n=0}^{N} (-CB^{-1})^n$$

both converge in norm and are bounded. Since $D = B^{-1}A$ the range of $D$ is contained in $D(B)$. So

$$(B + C)D = (B + C) \sum_{n=0}^{\infty} B^{-1}(-CB^{-1})^n = \sum_{n=0}^{\infty} (-CB^{-1})^n + \sum_{n=0}^{\infty} CB^{-1}(-CB^{-1})^n = 1$$

As $1$ is onto, $(B + C)$ must be onto as well. Since

$$D = \sum_{n=0}^{\infty} B^{-1}(-CB^{-1})^n = \sum_{n=0}^{\infty} (-B^{-1}C)^n B^{-1}$$

we also have $D(B + C) = 1$, just as above. Since $1$ is 1–1, $B + C$ must be 1–1 too. The inverse of $B + C$ is $D$, which is bounded. Furthermore

$$\|(B + C)^{-1} - B^{-1}\| = \left\| \sum_{n=1}^{\infty} B^{-1}(-CB^{-1})^n \right\| \leq \sum_{n=1}^{\infty} \|B^{-1}\|^{n+1}\|C\|^n = \frac{\|B^{-1}\|^2 \|C\|}{1 - \|B^{-1}\| \|C\|}$$

(b) is true. Let $\lambda, \mu \in \rho(A)$. We again have the resolvent formula

$$(\mu \mathbb{I} - A)^{-1} - (\lambda \mathbb{I} - A)^{-1} = (\mu \mathbb{I} - A)^{-1}[(\lambda \mathbb{I} - A) - (\mu \mathbb{I} - A)](\lambda \mathbb{I} - A)^{-1} = -(\mu - \lambda)(\mu \mathbb{I} - A)^{-1}(\lambda \mathbb{I} - A)^{-1}$$

(Note that the range of $(\lambda \mathbb{I} - A)^{-1}$ is the domain of $A$ and hence the domain of the operator in $[\cdots]$.) Hence

$$\frac{1}{\mu - \lambda}[(\mu \mathbb{I} - A)^{-1} - (\lambda \mathbb{I} - A)^{-1}] = -(\mu \mathbb{I} - A)^{-1}(\lambda \mathbb{I} - A)^{-1}$$

As $\mu \to \lambda$, the operator $(\mu \mathbb{I} - A)^{-1}$ converges in norm to $(\lambda \mathbb{I} - A)^{-1}$, so that the right hand side converges in norm to $-(\mathbb{I} - A)^{-2}$.

(c) is false. We showed in class that the operator $T = i\frac{d}{dx}$ on the domain $\{\varphi \in AC[0,1] \mid \varphi(0) = 0\}$ has $\sigma(T) = \emptyset$. When $T$ is a bounded operator, the argument that $\sigma(T) \neq \emptyset$ proceeds as follows. Suppose that $\rho(T) = \mathbb{C}$. Then $R_T(T) = (\lambda \mathbb{I} - T)^{-1}$ is analytic on the entire complex plane. Furthermore $\|R_T(T)\| \to 0$ as $|\lambda| \to \infty$. This is the step that fails when $T$ is not bounded. It was proven by expanding $(\lambda \mathbb{I} - T)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{T}{\lambda}\right)^n$. When $T$ is bounded this converges in norm for all $|\lambda| > \|T\|$. When $T$ is not bounded it doesn’t.