MATH 511 Problem Set 4 Solutions

1. Denote by $\mathcal{B}_{\mathbb{R}}$ the $\sigma$-algebra of Borel subsets of $\mathbb{R}$ and by $\mathcal{L}(\mathcal{H})$ the set of a bounded operators on a Hilbert space $\mathcal{H}$. Let the map $E: \mathcal{B}_{\mathbb{R}} \to \mathcal{L}(\mathcal{H})$ obey the following conditions.
   (i) For each $B \in \mathcal{B}_{\mathbb{R}}$, the operator $E(B)$ is an orthogonal projection on some closed subspace of $\mathcal{H}$.
   That is, $E(B)^2 = E(B)$ and $E(B) = E(B)^*$.
   (ii) If $A, B \in \mathcal{B}_{\mathbb{R}}$ are disjoint, then $E(A \cup B) = E(A) + E(B)$.

   **Solution.** Let $C = A \cap B$, $\tilde{A} = A \setminus C$ and $\tilde{B} = B \setminus C$. Then $\tilde{A}$, $\tilde{B}$ and $C$ are disjoint Borel subsets of $\mathbb{R}$ and
   
   \[ A = \tilde{A} \cup C \quad B = \tilde{B} \cup C \quad A \cup B = \tilde{A} \cup C \cup \tilde{B} \]

   Hence,
   
   \[
   E(A)E(B) = [E(\tilde{A}) + E(C)] [E(\tilde{B}) + E(C)]
   = E(\tilde{A})E(\tilde{B}) + E(\tilde{A})E(C) + E(C)E(\tilde{B}) + E(C)^2
   \]

   The first three terms are each zero by Problem Set 2, #3 with $(E, F) = (E(\tilde{A}), E(\tilde{B}))$ for the first term, $(E, F) = (E(\tilde{A}), E(C))$ for the second term and $(E, F) = (E(C), E(\tilde{B}))$ for the third term. As $E(C)$ is an orthogonal projection, we have $E(C)^2 = E(C)$ and hence $E(A)E(B) = E(A \cap B)$ as desired.

2. Let $E_A$ be the bounded projection valued measure associated to a self–adjoint operator $A$ on a Hilbert space $\mathcal{H}$.
   (a) Prove that
   
   \[
   \sigma(A) = \{ \lambda \in \mathbb{R} \mid E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0 \text{ for all } \varepsilon > 0 \}
   \]
   \[
   \sigma_p(A) = \{ \lambda \in \mathbb{R} \mid E_A(\{\lambda\}) \neq 0 \}
   \]
   \[
   E_A(\rho(A) \cap \mathbb{R}) = 0
   \]
   \[
   \text{range } E_A(\{0\}) = \ker A
   \]
   (b) Prove that, for all $-\infty < a < b < \infty$,
   \[
   \frac{1}{2} \{ E_A([a, b]) + E_A((a, b]) \} = \text{s-lim}_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{a}^{b} \{(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}\} d\lambda
   \]
   The integral on the right hand side is defined to be $f_\varepsilon(A)$ with $f_\varepsilon(x) = \int_{a}^{b} \left\{ \frac{1}{x - \lambda - i\varepsilon} - \frac{1}{x - \lambda + i\varepsilon} \right\} d\lambda$.
   (c) Let $-\infty < a < b < \infty$ with $a$ and $b$ being in the resolvent set of $A$. Denote by $R_\zeta(A) = (\zeta I - A)^{-1}$ the resolvent operator for $A$. Let $C_{a,b}$ be any counterclockwise oriented simple closed curve in the complex plane, for which the intersection of the interior of $C_{a,b}$ with the real line is the interval $(a, b)$. Prove that
   \[
   E_A([a, b]) = E_A([a, b]) = E_A((a, b]) = E_A((a, b]) = \frac{1}{2\pi i} \int_{C_{a,b}} R_\zeta(A) d\zeta
   \]

   **Solution.** (a) Second equation:
   
   \[
   \lambda \in \sigma_p(A) \iff \exists 0 \neq \psi \in \mathcal{H} \text{ s.t. } \langle \psi, (A - \lambda I)^2 \psi \rangle = \| (A - \lambda I) \psi \|^2 = 0
   \]
   \[
   \iff \exists 0 \neq \psi \in \mathcal{H} \text{ s.t. } \int |\mu - \lambda|^2 \langle \psi, E_A(\mu) \psi \rangle = 0
   \]
   \[
   \iff \exists 0 \neq \psi \in \mathcal{H} \text{ s.t. } \langle \psi, E_A(\mathbb{R} \setminus \{\lambda\}) \psi \rangle = 0
   \]
   \[
   \iff \exists 0 \neq \psi \in \mathcal{H} \text{ s.t. } \langle \psi, E_A(\{\lambda\}) \psi \rangle = \langle \psi, (I - E_A(\mathbb{R} \setminus \{\lambda\})) \psi \rangle = \| \psi \|^2
   \]
   \[
   \iff E_A(\{\lambda\}) \neq 0
   \]
(a) **First equation:** If $E_A(\{\lambda\}) \neq 0$ then certainly $E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$ for all $\varepsilon > 0$. So it suffices to consider $\lambda \notin \sigma_p(A)$.

- if, for some $\varepsilon > 0$, we have $E_A((\lambda - \varepsilon, \lambda + \varepsilon)) = 0$, then for any $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, 
  
  $$\| (A - \lambda \mathbb{I}) \psi \|^2 = \int |\mu - \lambda|^2 \, d\langle \psi, E_A(\mu)\psi \rangle \geq \int \varepsilon^2 \, d\langle \psi, E_A(\mu)\psi \rangle = \varepsilon^2$$

  so that $\lambda \in \rho(A)$.

- On the other hand, suppose that, for every $\varepsilon > 0$, we have $E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$. Fix any $\varepsilon > 0$. There is a $\psi \in \mathcal{H}$ of norm one in the range of $E_A((\lambda - \varepsilon, \lambda + \varepsilon))$. For such a $\psi$, $E_A(\Sigma)\psi = E_a(\Sigma)E_A((\lambda - \varepsilon, \lambda + \varepsilon))\psi = 0$, for any Borel $\Sigma \subset \mathbb{R}$ that does not intersect $(\lambda - \varepsilon, \lambda + \varepsilon)$. So 

  $$\| (A - \lambda \mathbb{I}) \psi \|^2 = \int |\mu - \lambda|^2 \, d\langle \psi, E_A(\mu)\psi \rangle \leq \int \varepsilon^2 \, d\langle \psi, E_A(\mu)\psi \rangle = \varepsilon^2$$

  Consequently $\lambda \notin \rho(A)$.

(b) **Third equation:** Set $S = \rho(A) \cap \mathbb{R}$. Let $\psi \in \mathcal{H}$ and let $\mu_\psi$ be the measure on $I = [-\|A\|, \|A\|]$ that obeys $\langle \psi, f(A)\psi \rangle = \int_I f(\lambda) \, d\mu_\psi(\lambda)$ for all $f \in C(I)$. We showed in Problem Set 3, #4 that $\mu_\psi(I \cap \rho(A)) = 0$. Since $S$ is an open subset of $\mathbb{R}$ the functions 

  $$f_n(\lambda) = \begin{cases} 0 & \text{if } \lambda \notin S \\ nd(\lambda, \mathbb{R} \setminus S) & \text{if } \lambda \in S \text{ and } d(\lambda, \mathbb{R} \setminus S) \leq \frac{1}{n} \\ 1 & \text{if } \lambda \in S \text{ and } d(\lambda, \mathbb{R} \setminus S) > \frac{1}{n} \end{cases}$$

  form a sequence of continuous functions that converge pointwise to $\chi_S$ and are uniformly bounded. By definition $E_A(S) = \chi_S(A)$. By the functional calculus version of the spectral theorem, $\chi_S(A)$ is the strong limit of the $f_n(A)$’s. Hence 

  $$\| E_A(S)\psi \|^2 = \langle \psi, E_A(S)\psi \rangle = \lim_{n \to \infty} \langle \psi, f_n(A)\psi \rangle = \lim_{n \to \infty} \int_I f_n(\lambda) \, d\mu_\psi(\lambda) = 0$$

  since $f_n$ is zero a.e. with respect to $\mu_\psi$. This is true for all $\psi \in \mathcal{H}$, so $E_A(S) = 0$.

(b) **Fourth equation:**

$$\psi \in \ker (A) \iff \| A\psi \|^2 = 0$$

$$\iff \int \mu^2 \, d\langle \psi, E_A(\mu)\psi \rangle = 0$$

$$\iff \langle \psi, E_A(\mathbb{R} \setminus \{0\})\psi \rangle = 0$$

$$\iff (\mathbb{1} - E_A(\{0\}))\psi = E_A(\mathbb{R} \setminus \{0\})\psi = 0$$

$$\iff \psi = E_A(\{0\})\psi$$

$$\iff \psi \in \text{range } E_A(\{0\})$$

(b) Observe that 

$$f_\varepsilon(x) = \int_a^b \left\{ \frac{1}{x - \lambda + \varepsilon} - \frac{1}{x - \lambda - \varepsilon} \right\} d\lambda = \int_a^b \frac{2i\varepsilon}{(x - \lambda)^2 + \varepsilon^2} d\lambda = 2i \tan^{-1} \left( \frac{\lambda - x}{\varepsilon} \right) \big|_{\lambda=a}^{\lambda=b}$$

is bounded uniformly in $x$ and $\varepsilon$ and converges pointwise in $x$ to 

$$2\pi i \left\{ \begin{array}{ll} 1 & \text{if } a < x < b \\ \frac{i}{2} & \text{if } x = a,b \\ 0 & \text{if } x < a \text{ or } x > b \end{array} \right\} = 2\pi i \left\{ \frac{i}{2} \chi_{[a,b]}(x) + \frac{1}{2} \chi_{(a,b)}(x) \right\}$$
as $\varepsilon \to 0$. Consequently, as $\varepsilon \to \infty$, the operator $\frac{1}{\varepsilon} f_\varepsilon(A)$ converges strongly to the operator $\frac{1}{2} \chi_{[a,b]}(A) + \frac{1}{2} \chi_{(a,b)}(A) = \frac{1}{2} E_A([a,b]) + \frac{1}{2} E_A((a,b))$, by the functional calculus version of the spectral theorem. 

(c) Fix any $\psi, \varphi \in H$. Then

$$\frac{1}{2\pi} \int_{C_{a,b}} \langle \psi, R_\lambda(A) \varphi \rangle \ d\lambda = \frac{1}{2\pi} \int_{C_{a,b}} \left[ \int \frac{1}{\lambda} \ d\langle \psi, E_\lambda(A) \varphi \rangle \right] \ d\lambda$$

By the third equation of part (a), we may restrict the domain of integration for $\lambda$ to $\sigma(A)$. As $\sigma(A)$ is a closed set that does not intersect the closed set $C_{a,b}$, the distance between the two is strictly larger than zero. Hence $\frac{1}{\lambda}$ is bounded on the domain of integration and we may interchange of order of integration is by Fubini. So

$$\frac{1}{2\pi} \int_{C_{a,b}} \langle \psi, R_\lambda(A) \varphi \rangle \ d\lambda = \int \left[ \frac{1}{2\pi} \int_{C_{a,b}} \frac{1}{\lambda} \ d\langle \psi, E_\lambda(A) \varphi \rangle \right] \ d\lambda$$

Now $\frac{1}{2\pi} \int_{C_{a,b}} \frac{1}{\lambda} \ d\lambda$ is one if $\lambda$ is inside $C_{a,b}$ and is zero if $\lambda$ is outside $C_{a,b}$. So

$$\frac{1}{2\pi} \int_{C_{a,b}} \langle \psi, R_\lambda(A) \varphi \rangle \ d\lambda = \int \chi_{[a,b]}(\lambda) \ d\langle \psi, E_\lambda(A) \varphi \rangle = \langle \psi, E_A([a,b]) \varphi \rangle$$

Since $\psi$ and $\varphi$ were arbitrary, we now have that

$$E_A([a,b]) = \frac{1}{2\pi} \int_{C_{a,b}} R_\lambda(A) \ d\lambda$$

That $E_A([a,b]) = E_A([a,b]) = E_A((a,b))$ is an immediate consequence of $E_A([a]) = E_A([b]) = 0$, which in turn is an immediate consequence of $E_A(\rho(A) \cap \mathbb{I}) = 0$.

3. Let $E$ be a bounded projection valued measure on a Hilbert space $H$. Denote by $B$ the set of all bounded Borel functions on $\mathbb{I}$ and define, for each $f \in B$,

$$\Phi(f) = \int f(\lambda) \ dE(\lambda)$$

(a) Prove that if $B$ is a Borel subset of $\mathbb{I}$, then $\Phi(\chi_B) = E(B)$.

(b) Let $\{f_n\}_{n \in \mathbb{N}} \subset B$ be a sequence of functions that converges pointwise on $\mathbb{I}$ to the function $f$.

Assume further that there is a constant $K$ such that $\sup_{x \in \mathbb{I}} |f_n(x)| \leq K$ for all $n \in \mathbb{N}$. Prove that

$$\Phi(f) = \lim_{n \to \infty} \Phi(f_n)$$

That is, prove that the operators $\Phi(f_n)$ converge weakly to $\Phi(f)$.

(c) Prove that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in B$.

**Solution.** (a) If $\varphi$ and $\psi$ are any two vectors in $H$, then, by definition

$$\langle \varphi, \Phi(\chi_B) \psi \rangle = \int \chi_B(\lambda) \ d\langle \varphi, E(\lambda) \psi \rangle = \langle \varphi, E(B) \psi \rangle$$

(b) We are to prove that $\lim_{n \to \infty} \langle \varphi, \Phi(f_n) \psi \rangle = \langle \varphi, \Phi(f) \psi \rangle$ for all $\varphi, \psi \in H$. By the polarization identity, it suffices to prove that $\lim_{n \to \infty} \langle \varphi, \Phi(f_n) \varphi \rangle = \langle \varphi, \Phi(f) \varphi \rangle$ for all $\varphi \in H$. So let $\varphi \in H$. Then

$$\lim_{n \to \infty} \langle \varphi, \Phi(f_n) \varphi \rangle = \lim_{n \to \infty} \int f_n(\lambda) \ d\langle \varphi, E(\lambda) \varphi \rangle = \int f(\lambda) \ d\langle \varphi, E(\lambda) \varphi \rangle = \langle \varphi, \Phi(f) \varphi \rangle$$

For the second equality, we used the Lebesgue dominated convergence theorem.
(c) **Step 1:** By part (a) and Problem 1, we have, for any Borel subsets $B, C$ of $\mathbb{R}$,

$$\Phi(\chi_B)\Phi(\chi_C) = E(B)E(C) = E(B \cap C) = \phi(\chi_{B \cap C}) = \Phi(\chi_B \chi_C)$$

(c) **Step 2:** As both $\Phi(f)\Phi(g)$ and $\Phi(fg)$ are linear in both $f$ and $g$, we have that $\Phi(f)\Phi(g) = \Phi(fg)$ for all simple functions $f, g \in B$.

(c) **Step 3:** Fix any vectors $\varphi, \psi \in \mathcal{H}$ and any simple function $g \in B$. Any function $f \in B$ is a uniformly bounded pointwise limit of a sequence $\{f_n\}_{n \in \mathbb{N}}$ of simple functions in $B$. So, by part (b),

$$\langle \varphi, \Phi(f)\Phi(g)\psi \rangle = \langle \varphi, \Phi(f) (\Phi(g)\psi) \rangle = \lim_{n \to \infty} \langle \varphi, \Phi(f_n) (\Phi(g)\psi) \rangle = \lim_{n \to \infty} \langle \varphi, \Phi(f_n g) \psi \rangle$$

So we now have $\Phi(f)\Phi(g) = \Phi(fg)$ for all $f \in B$ and all simple functions $g \in B$.

(c) **Step 4:** Fix any vectors $\varphi, \psi \in \mathcal{H}$ and any function $f \in B$. Any function $g \in B$ is a uniformly bounded pointwise limit of a sequence $\{g_n\}_{n \in \mathbb{N}}$ of simple functions in $B$. So, by part (b),

$$\langle \varphi, \Phi(f)\Phi(g)\psi \rangle = \langle \Phi(f)^*\varphi, \Phi(g)\psi \rangle = \lim_{n \to \infty} \langle \Phi(f)^*\varphi, \Phi(g_n)\psi \rangle = \lim_{n \to \infty} \langle \varphi, \Phi(f)\Phi(g_n)\psi \rangle$$

So we now have $\Phi(f)\Phi(g) = \Phi(fg)$ for all $f, g \in B$.

4. Let $\mathcal{H}$ be a Hilbert space and $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ be a sequence of orthogonal projections that obey $E_mE_n = 0$ for all $m, n \in \mathbb{N}$ with $m \neq n$. Prove that $\sum_{n=1}^{\infty} E_n$ converges strongly to an orthogonal projection.

**Solution.** Let $\varphi \in \mathcal{H}$. Define, for each $n \in \mathbb{N}$, $P_n = \sum_{m=1}^{n} E_m$. By Problem Set 2 #3, $P_n$ is again an orthogonal projection, so that

$$\|\varphi\|^2 \geq \|P_n\varphi\|^2 = \sum_{\ell,m=1}^{n} \langle E_{\ell}\varphi, E_m\varphi \rangle = \sum_{\ell=1}^{n} \langle E_{\ell}\varphi, E_{\ell}\varphi \rangle = \sum_{\ell=1}^{n} \|E_{\ell}\varphi\|^2$$

Consequently, the series $\sum_{n=1}^{\infty} \|E_{\ell}\varphi\|^2$ converges. Similarly, for any $m, n \in \mathbb{N}$ with $n > m$,

$$\|P_n\varphi - P_m\varphi\|^2 = \sum_{\ell=m+1}^{n} \|E_{\ell}\varphi\|^2$$

Hence the sequence $\{P_n\varphi\}_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{H}$ and so converges to some $P\varphi$. Since every $P_m$ is linear, self-adjoint and of norm at most 1, the same is true for $P$. For every $n \in \mathbb{N}$ and $\varphi \in \mathcal{H}$,

$$\|P^2\varphi - P\varphi\| = \|P^2\varphi - P_n^2\varphi + P_n\varphi - P\varphi\|$$

$$\leq \|P(P\varphi) - P_n(P\varphi)\| + \|P_n(P\varphi - P_n\varphi)\| + \|P_n\varphi - P\varphi\|$$

$$\leq \|P(P\varphi) - P_n(P\varphi)\| + 2\|P_n\varphi - P\varphi\|$$

This converges to zero as $n \to \infty$, so $P^2 = P$ and $P$ is an orthogonal projection.

5. Let $A \in \mathcal{L}(\mathcal{H})$ and define $|A| = \sqrt{A^*A}$ with the square root defined in Problem Set 3 #6.d. Define $U \in \mathcal{L}(\mathcal{H})$ by applying the BLT theorem to

$$U \varphi = A\psi \quad \text{for } \varphi = |A|\psi \in \text{range } |A|$$

$$U \varphi = 0 \quad \text{for } \varphi \in (\text{range } |A|)^\perp$$
a) Prove that $U$ is well-defined and is a partial isometry. That is, $U$ is unitary between $(\ker U) \perp$ and range $U$.

b) Prove that $A = U|A|$. This is called the polar decomposition of $A$.

c) Prove that

$$U = \lim_{n \to \infty} A f_n(|A|)$$

where

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \geq \frac{1}{n} \\ 1 & \text{if } |x| \leq \frac{1}{n} \end{cases}$$

Solution. (a) Well-defined on range $|A|$: If $|A| \psi = |A| \psi$, then $|A| (\tilde{\psi} - \psi) = 0$ so that

$$0 = \| |A| (\tilde{\psi} - \psi) \|^2 = \langle \tilde{\psi} - \psi, |A|^2 (\tilde{\psi} - \psi) \rangle = \langle \tilde{\psi} - \psi, A^* A (\tilde{\psi} - \psi) \rangle = \| A (\tilde{\psi} - \psi) \|^2$$

and $A \tilde{\psi} = A \psi$.

(a) Partial isometry: If $\varphi = |A| \psi \in \text{range} \ |A|$, then

$$\| \varphi \|^2 = \| |A| \psi \|^2 = \langle |A| \psi, |A| \psi \rangle = \langle \psi, |A|^2 \psi \rangle = \langle \psi, A^* A \psi \rangle = \| A \psi \|^2 = \| U \varphi \|^2$$

Consequently $U$, defined as a linear operator on $\{ \varphi_1 + \varphi_2 \mid \varphi_1 \in \text{range} \ |A|, \ \varphi_2 \in (\text{range} \ |A|) \perp \}$, which is dense in $\mathcal{H}$ has operator norm at most one. By the BLT theorem, $U$ has a unique continuous extension to $\mathcal{H}$, which we also call $U$. By continuity, this $U$ also obeys $\| \varphi \|^2 = \| U \varphi \|^2$ for all $\varphi \in \text{range} \ |A|$ and so is unitary from range $|A|$ to range $U$.

If $\varphi_1 \in \text{range} \ |A|$ and $\varphi_2 \in (\text{range} \ |A|) \perp$, then

$$\| U (\varphi_1 + \varphi_2) \| = \| U \varphi_1 \| = \| \varphi_1 \|$$

so that $U (\varphi_1 + \varphi_2) = 0$ if and only if $\varphi_1 = 0$. Thus $\ker U = (\text{range} \ |A|) \perp$ and $(\ker U) \perp = \text{range} \ |A|$ and $U$ is a partial isometry.

(b) For all $\psi \in \mathcal{H}$, writing $\varphi = |A| \psi$, we have

$$U |A| \psi = U \varphi = A \psi$$

(c) Set, for each $n \in \mathbb{N}$,

$$g_n(x) = x f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \geq \frac{1}{n} \\ 1 & \text{if } |x| \leq \frac{1}{n} \end{cases}$$

Observe that $g_n(x)$ is continuous and converges pointwise to 1 as $n \to \infty$, except at $x = 0$ where it is 0. Furthermore $|g_n(x)| \leq 1$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Consequently, by the functional calculus version of the spectral theorem and the definition of $E_{|A|}$, $g_n(|A|)$ converges strongly to $\chi_{\mathbb{R} \setminus \{0\}}(|A|) = E_{|A|}(\mathbb{R} \setminus \{0\}) = I - E_{|A|}(\{0\})$. So, for all $\varphi \in \mathcal{H}$,

$$A f_n(|A|) \varphi = U |A| f_n(|A|) \varphi = U g_n(|A|) \varphi \overset{n \to \infty}{\to} U (I - E_{|A|}(\{0\})) \varphi = U \varphi$$

since the range of $E_{|A|}(\{0\})$ is the kernel of $|A|$ (by Problem #2a), which is the orthogonal complement of the range of $|A|$ (since $|A|$ is self-adjoint) which is contained in the kernel of $U$ (by the definition of $U$).