MATH 511 Problem Set 4 Solutions

1. Denote by $B_{\mathbb{R}}$ the $\sigma$–algebra of Borel subsets of $\mathbb{R}$ and by $L(H)$ the set of a bounded operators on a Hilbert space $H$. Let the map $E : B_{\mathbb{R}} \to L(H)$ obey the following conditions.
   
   (i) For each $B \in B_{\mathbb{R}}$, the operator $E(B)$ is an orthogonal projection on some closed subspace of $H$. That is, $E(B)^2 = E(B)$ and $E(B) = E(B)^*$.
   
   (ii) If $A, B \in B_{\mathbb{R}}$ are disjoint, then $E(A \cup B) = E(A) + E(B)$.

Solution. Let $C = A \cap B$, $\tilde{A} = A \setminus C$ and $\tilde{B} = B \setminus C$. Then $\tilde{A}$, $\tilde{B}$ and $C$ are disjoint Borel subsets of $\mathbb{R}$ and

$$A = \tilde{A} \cup C \quad B = \tilde{B} \cup C \quad A \cup B = \tilde{A} \cup C \cup \tilde{B}$$

Hence,

$$E(A)E(B) = [E(\tilde{A}) + E(C)] [E(\tilde{B}) + E(C)]$$

$$= E(\tilde{A})E(\tilde{B}) + E(\tilde{A})E(C) + E(C)E(\tilde{B}) + E(C)^2$$

The first three terms are each zero by Problem Set 2, #3 with $(E, F) = (E(\tilde{A}), E(\tilde{B}))$ for the first term, $(E, F) = (E(\tilde{A}), E(C))$ for the second term and $(E, F) = (E(C), E(\tilde{B}))$ for the third term. As $E(C)$ is an orthogonal projection, we have $E(C)^2 = E(C)$ and hence $E(A)E(B) = E(A \cap B)$ as desired.

2. Let $E_A$ be the bounded projection valued measure associated to a self–adjoint operator $A$ on a Hilbert space $H$.

(a) Prove that

$$\sigma(A) = \{ \lambda \in \mathbb{R} \mid E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0 \text{ for all } \varepsilon > 0 \}$$

$$\sigma_p(A) = \{ \lambda \in \mathbb{R} \mid E_A(\{\lambda\}) \neq 0 \}$$

$$E_A(\rho(A) \cap \mathbb{R}) = 0$$

$$\text{range } E_A(\{0\}) = \ker A$$

(b) Prove that, for all $-\infty < a < b < \infty$,

$$\frac{1}{2} \left\{ E_A([a, b]) + E_A((a, b]) \right\} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b \{(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}\} \, d\lambda$$

The integral on the right hand side is defined to be $f_\varepsilon(A)$ with $f_\varepsilon(x) = \int_a^b \left\{ \frac{1}{z - \lambda - i\varepsilon} - \frac{1}{z - \lambda + i\varepsilon} \right\} \, d\lambda$.

(c) Let $-\infty < a < b < \infty$ with $a$ and $b$ being in the resolvent set of $A$. Denote by $R_\zeta(A) = (\zeta I - A)^{-1}$ the resolvent operator for $A$. Let $C_{a,b}$ be any counterclockwise oriented simple closed curve in the complex plane, for which the intersection of the interior of $C_{a,b}$ with the real line is the interval $(a, b)$. Prove that

$$E_A([a, b]) = E_A([a, b]) = E_A((a, b]) = E_A((a, b]) = \frac{1}{2\pi i} \int_{C_{a,b}} R_\zeta(A) \, d\zeta$$

Solution. (a) Second equation:

$$\lambda \in \sigma_p(A) \iff \exists \varepsilon > 0 \neq \psi \in H \quad \text{s.t.} \quad \langle \psi, (A - \lambda I)^2 \psi \rangle = \| (A - \lambda I) \psi \|^2 = 0$$

$$\iff \exists \varepsilon > 0 \neq \psi \in H \quad \text{s.t.} \quad \int |\mu - \lambda|^2 \, d\langle \psi, E_A(\mu) \psi \rangle = 0$$

$$\iff \exists \varepsilon > 0 \neq \psi \in H \quad \text{s.t.} \quad \langle \psi, E_A(\mu) \psi \rangle = 0$$

$$\iff \exists \varepsilon > 0 \neq \psi \in H \quad \text{s.t.} \quad \langle \psi, E_A(\{\lambda\}) \psi \rangle = 0$$

$$\iff \exists \varepsilon > 0 \neq \psi \in H \quad \text{s.t.} \quad \langle \psi, (\mathbb{I} - E_A(\{\lambda\})) \psi \rangle = \| \psi \|^2$$

$$\iff E_A(\{\lambda\}) \neq 0$$
(a) First equation: If $E_A(\{\lambda\}) \neq 0$ then certainly $E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$ for all $\varepsilon > 0$. So it suffices to consider $\lambda \notin \sigma_p(A)$.

- if, for some $\varepsilon > 0$, we have $E_A((\lambda - \varepsilon, \lambda + \varepsilon)) = 0$, then for any $\psi \in \mathcal{H}$ with $\|\psi\| = 1$,

\[ \|A - \lambda I\|\psi\|^2 = \int |\mu - \lambda|^2 d\langle \psi, E_A(\mu)\psi \rangle \geq \int \varepsilon^2 d\langle \psi, E_A(\mu)\psi \rangle = \varepsilon^2 \]

so that $\lambda \in \rho(A)$.

- On the other hand, suppose that, for every $\varepsilon > 0$, we have $E_A((\lambda - \varepsilon, \lambda + \varepsilon)) \neq 0$. Fix any $\varepsilon > 0$.

There is a $\psi \in \mathcal{H}$ of norm one in the range of $E_A((\lambda - \varepsilon, \lambda + \varepsilon))$. For such a $\psi$, $E_A(\Sigma)\psi = E_a(\Sigma)E_A((\lambda - \varepsilon, \lambda + \varepsilon))\psi = 0$, for any Borel $\Sigma \subset \mathbb{R}$ that does not intersect $(\lambda - \varepsilon, \lambda + \varepsilon)$. So

\[ \|A - \lambda I\|\psi\|^2 = \int |\mu - \lambda|^2 d\langle \psi, E_A(\mu)\psi \rangle \leq \int \varepsilon^2 d\langle \psi, E_A(\mu)\psi \rangle = \varepsilon^2 \]

Consequently $\lambda \notin \rho(A)$.

(a) Third equation: Set $S = \rho(A) \cap \mathbb{R}$. Let $\psi \in \mathcal{H}$ and let $\mu_\psi$ be the measure on $I = [-\|A\|, \|A\|]$ that obeys $\langle \psi, f(A)\psi \rangle = \int_I f(\lambda) d\mu_\psi(\lambda)$ for all $f \in C(I)$. We showed in Problem Set 3, #4 that $\mu_\psi(I \cap \rho(A)) = 0$. Since $S$ is an open subset of $\mathbb{R}$ the functions

\[ f_n(\lambda) = \begin{cases} 
0 & \text{if } \lambda \notin S \\
nD(\lambda, \mathbb{R} \setminus S) & \text{if } \lambda \in S \text{ and } d(\lambda, \mathbb{R} \setminus S) \leq \frac{1}{n} \\
1 & \text{if } \lambda \in S \text{ and } d(\lambda, \mathbb{R} \setminus S) > \frac{1}{n}
\end{cases} \]

form a sequence of continuous functions that converge pointwise to $\chi_S$ and are uniformly bounded. By definition $E_A(S) = \chi_S(A)$. By the functional calculus version of the spectral theorem, $\chi_S(A)$ is the strong limit of the $f_n(A)$’s. Hence

\[ \|E_A(S)\psi\|^2 = \langle \psi, E_A(S)\psi \rangle = \lim_{n \to \infty} \langle \psi, f_n(A)\psi \rangle = \lim_{n \to \infty} \int_I f_n(\lambda) d\mu_\psi(\lambda) = 0 \]

since $f_n$ is zero a.e. with respect to $\mu_\psi$. This is true for all $\psi \in \mathcal{H}$, so $E_A(S) = 0$.

(a) Fourth equation:

\[ \psi \in \ker(A) \iff \|A\psi\|^2 = 0 \]

\[ \iff \int \mu^2 d\langle \psi, E_A(\mu)\psi \rangle = 0 \]

\[ \iff \langle \psi, E_A(\mathbb{R} \setminus \{0\})\psi \rangle = 0 \]

\[ \iff (I - E_A(\{0\}))\psi = E_A(\mathbb{R} \setminus \{0\})\psi = 0 \]

\[ \iff E_A(\{0\})\psi = \psi \]

\[ \iff \psi \in \text{range} \ E_A(\{0\}) \]

(b) Observe that

\[ f_\varepsilon(x) = \int_a^b \left\{ \frac{1}{x-\lambda - \varepsilon} - \frac{1}{x-\lambda + \varepsilon} \right\} d\lambda = \int_a^b \frac{2\varepsilon}{(x-\lambda)^2 + \varepsilon^2} d\lambda = 2i \tan^{-1} \left( \frac{\lambda - x}{\varepsilon} \right) \big|_{\lambda = a}^{\lambda = b} \]

is bounded uniformly in $x$ and $\varepsilon$ and converges pointwise in $x$ to \[ 2\pi i \left\{ \begin{array}{ll} 1 & \text{if } a < x < b \\ \frac{1}{2} & \text{if } x = a, b \\ 0 & \text{if } x < a \text{ or } x > b \end{array} \right. \]

\[ = 2\pi i \left\{ \frac{1}{2} \chi_{[a,b]}(x) + \frac{1}{2} \chi_{(a,b)}(x) \right\} \]
as \( \varepsilon \to 0 \). Consequently, as \( \varepsilon \to \infty \), the operator \( \frac{1}{2\pi i}f(\varepsilon) \) converges strongly to the operator \( \frac{1}{2}\chi_{[a,b]}(A) + \frac{1}{2}\chi_{(a,b)}(A) = \frac{1}{2}E_A([a,b]) + \frac{1}{2}E_A((a,b)) \), by the functional calculus version of the spectral theorem.

(c) Fix any \( \psi, \varphi \in \mathcal{H} \). Then

\[
\frac{1}{2\pi i} \int_{C_{a,b}} \langle \psi, R_\zeta(A)\varphi \rangle \ d\zeta = \frac{1}{2\pi i} \int_{C_{a,b}} \left[ \int \frac{1}{\zeta - \lambda} \ d \langle \psi, E_A(\lambda)\varphi \rangle \right] d\zeta
\]

By the third equation of part (a), we may restrict the domain of integration for \( \lambda \) to \( \sigma(A) \). As \( \sigma(A) \) is a closed set that does not intersect the closed set \( C_{a,b} \), the distance between the two is strictly larger than zero. Hence \( \frac{1}{|\zeta - \lambda|} \) is bounded on the domain of integration and we may interchange order of integration is by Fubini. So

\[
\frac{1}{2\pi i} \int_{C_{a,b}} \langle \psi, R_\zeta(A)\varphi \rangle \ d\zeta = \int \frac{1}{2\pi i} \int_{C_{a,b}} \frac{1}{\zeta - \lambda} \ d \langle \psi, E_A(\lambda)\varphi \rangle
\]

Now \( \frac{1}{2\pi i} \int_{C_{a,b}} \frac{1}{\zeta - \lambda} \ d\zeta \) is one if \( \lambda \) is inside \( C_{a,b} \) and is zero if \( \lambda \) is outside \( C_{a,b} \). So

\[
\frac{1}{2\pi i} \int_{C_{a,b}} \langle \psi, R_\zeta(A)\varphi \rangle \ d\zeta = \int \chi_{(a,b)}(\lambda) \ d \langle \psi, E_A(\lambda)\varphi \rangle
\]

Since \( \psi \) and \( \varphi \) were arbitrary, we now have that

\[
E_A((a,b)) = \frac{1}{2\pi i} \int_{C_{a,b}} R_\zeta(A) \ d\zeta
\]

That \( E_A([a,b]) = E_A([a,b]) = E_A((a,b)) = E_A((a,b)) \) is an immediate consequence of \( E_A([a]) = E_A([b]) = 0 \), which in turn is an immediate consequence of \( E_A(\rho(A) \cap \mathbb{R}) = 0 \).

3. Let \( E \) be a bounded projection valued measure on a Hilbert space \( \mathcal{H} \). Denote by \( \mathcal{B} \) the set of all bounded Borel functions on \( \mathbb{R} \) and define, for each \( f \in \mathcal{B} \),

\[
\Phi(f) = \int f(\lambda) \ dE(\lambda)
\]

(a) Prove that if \( B \) is a Borel subset of \( \mathbb{R} \), then \( \Phi(\chi_B) = E(B) \).

(b) Let \( \{f_n\}_{n \in \mathbb{N}} \subset \mathcal{B} \) be a sequence of functions that converges pointwise on \( \mathbb{R} \) to the function \( f \).

Assume further that there is a constant \( K \) such that \( \sup_{x \in \mathbb{R}} |f_n(x)| \leq K \) for all \( n \in \mathbb{N} \). Prove that

\[
\Phi(f) = \text{w-lim}_{n \to \infty} \Phi(f_n)
\]

That is, prove that the operators \( \Phi(f_n) \) converge weakly to \( \Phi(f) \).

(c) Prove that \( \Phi(fg) = \Phi(f)\Phi(g) \) for all \( f, g \in \mathcal{B} \).

Solution. (a) If \( \varphi \) and \( \psi \) are any two vectors in \( \mathcal{H} \), then, by definition

\[
\langle \varphi, \Phi(\chi_B) \psi \rangle = \int \chi_B(\lambda) \ d \langle \varphi, E(\lambda) \psi \rangle = \langle \varphi, E(B) \psi \rangle
\]

(b) If \( \varphi \) and \( \psi \) are any two vectors in \( \mathcal{H} \), then

\[
\lim_{n \to \infty} \langle \varphi, \Phi(f_n) \psi \rangle = \lim_{n \to \infty} \int f_n(\lambda) \ d \langle \varphi, E(\lambda) \psi \rangle = \int f(\lambda) \ d \langle \varphi, E(\lambda) \psi \rangle = \langle \varphi, \Phi(f) \ psi \rangle
\]
For the second equality, we used the Lebesgue dominated convergence theorem.

(c) Step 1: By part (a) and Problem 1, we have, for any Borel subsets $B, C$ of $\mathbb{R}$,

$$
\Phi(\chi_B)\Phi(\chi_C) = E(B)E(C) = E(B \cap C) = \phi(\chi_{B \cap C}) = \Phi(\chi_B \chi_C)
$$

(c) Step 2: As both $\Phi(f)\Phi(g)$ and $\Phi(fg)$ are linear in both $f$ and $g$, we have that $\Phi(f)\Phi(g) = \Phi(fg)$ for all simple functions $f, g \in \mathcal{B}$.

(c) Step 3: Fix any vectors $\varphi, \psi \in \mathcal{H}$ and any simple function $g \in \mathcal{B}$. Any function $f \in \mathcal{B}$ is a uniformly bounded pointwise limit of a sequence $\{f_n\}_{n \in \mathbb{N}}$ of simple functions in $\mathcal{B}$. So, by part (b),

$$
\langle \varphi, \Phi(f)(g)\psi \rangle = \langle \varphi, \Phi(f) (\Phi(g)\psi) \rangle = \lim_{n \to \infty} \langle \varphi, \Phi(f_n) (\Phi(g)\psi) \rangle = \lim_{n \to \infty} \langle \varphi, \Phi(f_n g)\psi \rangle
$$

So we now have $\Phi(f)\Phi(g) = \Phi(fg)$ for all $f, g \in \mathcal{B}$.

(c) Step 4: Fix any vectors $\varphi, \psi \in \mathcal{H}$ and any function $f \in \mathcal{B}$. Any function $g \in \mathcal{B}$ is a uniformly bounded pointwise limit of a sequence $\{g_n\}_{n \in \mathbb{N}}$ of simple functions in $\mathcal{B}$. So, by part (b),

$$
\langle \varphi, \Phi(f)(g)\psi \rangle = \langle \varphi, \Phi(f) (\Phi(g)\psi) \rangle = \lim_{n \to \infty} \langle \varphi, \Phi(f_n) (\Phi(g)\psi) \rangle = \lim_{n \to \infty} \langle \varphi, \Phi(f_n g)\psi \rangle
$$

So we now have $\Phi(f)\Phi(g) = \Phi(fg)$ for all $f, g \in \mathcal{B}$.

4. Let $\mathcal{H}$ be a Hilbert space and $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ be a sequence of orthogonal projections that obey $E_mE_n = 0$ for all $m, n \in \mathbb{N}$ with $m \neq n$. Prove that $\sum_{n=1}^{\infty} E_n$ converges strongly to an orthogonal projection.

**Solution.** Let $\varphi \in \mathcal{H}$. Define, for each $n \in \mathbb{N}$, $P_n = \sum_{m=1}^{n} E_m$. By Problem Set 2 #3, $P_n$ is again an orthogonal projection, so that

$$
\|\varphi\|^2 \geq \|P_n\varphi\|^2 = \sum_{\ell, m=1}^{n} \langle E_{\ell}\varphi, E_m\varphi \rangle = \sum_{\ell=1}^{n} \langle E_{\ell}\varphi, E_{\ell}\varphi \rangle = \sum_{\ell=1}^{n} \|E_{\ell}\varphi\|^2
$$

Consequently, the series $\sum_{n=1}^{\infty} \|E_{\ell}\varphi\|^2$ converges. Similarly, for any $m, n \in \mathbb{N}$ with $n > m$,

$$
\|P_n\varphi - P_m\varphi\|^2 = \sum_{\ell=m+1}^{n} \|E_{\ell}\varphi\|^2
$$

Hence the sequence $\{P_n\varphi\}_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{H}$ and so converges to some $P\varphi$. Since every $P_m$ is linear, self–adjoint and of norm at most 1, the same is true for $P$. For every $n \in \mathbb{N}$ and $\varphi \in \mathcal{H},$

$$
\|P^2\varphi - P\varphi\| = \|P^2\varphi - P_n^2\varphi + P_n\varphi - P\varphi\|
$$

$$
\leq \|P(P\varphi) - P_n(P\varphi)\| + \|P_n(P\varphi - P_n\varphi)\| + \|P_n\varphi - P\varphi\|
$$

$$
\leq \|P(P\varphi) - P_n(P\varphi)\| + 2\|P_n\varphi - P\varphi\|
$$

This converges to zero as $n \to \infty$, so $P^2 = P$ and $P$ is an orthogonal projection.

5. Let $A \in \mathcal{L}(\mathcal{H})$ and define $|A| = \sqrt{A^*A}$ with the square root defined in Problem Set 3 #6.d. Define $U \in \mathcal{L}(\mathcal{H})$ by applying the BLT theorem to

$$
U\varphi = A\psi \quad \text{for } \varphi = |A|\psi \in \text{range } |A|
$$

$$
U\varphi = 0 \quad \text{for } \varphi \in (\text{range } |A|)^\perp
$$
a) Prove that $U$ is well–defined and is a partial isometry. That is, $U$ is unitary between $(\ker U)^\perp$ and range $U$.
b) Prove that $A = U|A|$. This is called the polar decomposition of $A$.
c) Prove that

$$U = \operatorname{s-lim}_{n \to \infty} A f_n(|A|)$$

where

$$f_n(x) = \begin{cases} \frac{1}{x} & \text{if } |x| \geq \frac{1}{n} \\ \frac{x}{n} & \text{if } |x| \leq \frac{1}{n} \end{cases}$$

**Solution.** (a) Well–defined on range $|A|$: If $|A|\varphi = |A|\psi$, then $|A|(\varphi - \psi) = 0$ so that

$$0 = |||A|(\varphi - \psi)||^2 = \langle \varphi - \psi, |A|^2(\varphi - \psi) \rangle = \langle \varphi - \psi, A^*A(\varphi - \psi) \rangle = ||A(\varphi - \psi)||^2$$

and $A\varphi = A\psi$.

(a) Partial isometry: If $\varphi = |A|\psi \in \text{range } |A|$, then

$$||\varphi||^2 = |||A|\psi||^2 = \langle |A|\psi, |A|\psi \rangle = \langle \psi, |A|^2\psi \rangle = \langle \psi, A^*A\psi \rangle = ||A\psi||^2 = ||U\varphi||^2$$

Consequently $U$, defined as a linear operator on $\{ \varphi_1 + \varphi_2 \mid \varphi_1 \in \text{range } |A|, \ \varphi_2 \in (\text{range } |A|)^\perp \}$, which is dense in $\mathcal{H}$ has operator norm at most one. By the BLT theorem, $U$ has a unique continuous extension to $\mathcal{H}$, which we also call $U$. By continuity, this $U$ also obeys $||\varphi||^2 = ||U\varphi||^2$ for all $\varphi \in \text{range } |A|$ and so is unitary from range $|A|$ to range $U$.

If $\varphi_1 \in \text{range } |A|$ and $\varphi_2 \in (\text{range } |A|)^\perp$, then

$$||U(\varphi_1 + \varphi_2)|| = ||U\varphi_1|| = ||\varphi_1||$$

so that $U(\varphi_1 + \varphi_2) = 0$ if and only if $\varphi_1 = 0$. Thus $\ker U = (\text{range } |A|)^\perp$ and $(\ker U)^\perp = \text{range } |A|$ and $U$ is a partial isometry.

(b) For all $\psi \in \mathcal{H}$, writing $\varphi = |A|\psi$, we have

$$U|A|\psi = U\varphi = A\psi$$

(c) Set, for each $n \in \mathbb{N}$,

$$g_n(x) = xf_n(x) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{n} \\ \frac{x}{n} & \text{if } |x| \leq \frac{1}{n} \end{cases}$$

Observe that $g_n(x)$ is continuous and converges pointwise to 1 as $n \to \infty$, except at $x = 0$ where it is 0. Furthermore $|g_n(x)| \leq 1$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Consequently, by the functional calculus version of the spectral theorem and the definition of $E_{|A|}$, $g_n(|A|)$ converges strongly to $\chi_{\mathbb{R} \setminus \{0\}}(|A|) = E_{|A|}(\mathbb{R} \setminus \{0\}) = \mathbb{1} - E_{|A|}(\{0\})$. So, for all $\varphi \in \mathcal{H}$,

$$A f_n(|A|)\varphi = U|A|f_n(|A|)\varphi = U g_n(|A|)\varphi \xrightarrow{n \to \infty} U(\mathbb{1} - E_{|A|}(\{0\}))\varphi = U\varphi$$

since the range of $E_{|A|}(\{0\})$ is the kernel of $|A|$ (by Problem # 2a), which is the orthogonal complement of the range of $|A|$ (since $|A|$ is self–adjoint) which is contained in the kernel of $U$ (by the definition of $U$).