Another Riesz Representation Theorem

In these notes we prove (one version of) a theorem known as the Riesz Representation Theorem. Some people also call it the Riesz–Markov Theorem. It expresses positive linear functionals on $C(X)$ as integrals over $X$. For simplicity, we will here only consider the case that $X$ is a compact metric space. We denote the metric $d(x, y)$. For more general versions of the theorem see

- H. L. Royden, Real Analysis, Macmillan, chapter 13, sections 4 and 5.

The background definitions that we need are

**Definition 1** A map $\ell : C(X) \to \mathbb{C}$ is a positive linear functional if

(a) $\ell(\alpha \varphi + \beta \psi) = \alpha \ell(\varphi) + \beta \ell(\psi)$ for all $\alpha, \beta \in \mathbb{C}$ and all $\varphi, \psi \in C(X)$ and

(b) $\ell(\varphi) \geq 0$ for all $\varphi \in C(X)$ that obey $\varphi(x) \geq 0$ for all $x \in X$.

**Problem 1** Let $\ell : C(X) \to \mathbb{C}$ be a positive linear functional. Prove that

$$|\ell(\varphi)| \leq \ell(1) \|\varphi\|_{C(X)}$$

for all $\varphi \in C(X)$. Here 1 is of course the function on $X$ that always takes the value 1.

**Definition 2**

(a) The set, $\mathcal{B}_X$, of Borel subsets of $X$ is the smallest $\sigma$–algebra that contains all open subsets of $X$.

(b) A Borel measure on $X$ is a measure $\mu : \mathcal{B}_X \to [0, \infty]$.

(c) A Borel measure, $\mu$ on $X$ is said to be regular if

(i) $\mu(A) = \inf \{ \mu(O) \mid A \subset O, \ O \text{ open} \}$

(ii) $\mu(A) = \sup \{ \mu(C) \mid C \subset A, \ C \text{ compact} \}$

for all $A \in \mathcal{B}_X$.

We shall prove

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Theorem 3 (Riesz Representation) Let $X$ be a compact metric space. If $\ell : C(X) \to \mathbb{C}$ is a positive linear functional on $C(X)$, then there exists a unique regular Borel measure $\mu$ on $X$ such that

$$\ell(f) = \int f(x) \, d\mu(x)$$

The measure $\mu$ is finite.

By way of motivation for the proof, let's guess what the measure is. To do so, we assume that $\ell(f) = \int f(x) \, d\mu(x)$ and derive a formula for $\mu$ in terms of $\ell$. We start off by considering any open $O \subset X$. Of course $\mu(O) = \int_X \chi_O(x) \, d\mu(x)$. As $\chi_O$ is not continuous, we cannot express $\mu(O) = \ell(\chi_O)$. But we can express $\chi_O$ as a limit of continuous functions. For each $n \in \mathbb{N}$, set

$$f_n(x) = \begin{cases} 0 & \text{if } x \in X \setminus O \\ n \, d(x, X \setminus O) & \text{if } x \in O \text{ and } d(x, X \setminus O) \leq \frac{1}{n} \\ 1 & \text{if } x \in O \text{ and } d(x, X \setminus O) \geq \frac{1}{n} \end{cases}$$

This is a sequence of continuous functions on $X$ with $0 \leq f_n(x) \leq 1$ for all $n \in \mathbb{N}$ and $x \in X$. Because $O$ is open, $d(x, X \setminus O) > 0$ for all $x \in O$ and $\lim_{n \to \infty} f_n(x) = \chi_O(x)$ for all $x \in X$. So by the dominated convergence theorem (or, if you prefer, the monotone convergence theorem)

$$\mu(O) = \lim_{n \to \infty} \ell(f_n)$$

Of course, this only determines $\mu$ on open sets. But if $\mu$ is regular, it is completely determined by its values on opens sets. We are now ready to start the proof itself.

Define, for any open set $O \subset X$,

$$\mu^*(O) = \sup \left\{ \ell(f) \mid f \in C(X), \, f \upharpoonright X \setminus O = 0, \, 0 \leq f(x) \leq 1 \text{ for all } x \in X \right\}$$

and, for any $A \subset X$,

$$\mu^*(A) = \inf \left\{ \mu^*(O) \mid O \subset X, \, O \text{ open }, \, A \subset O \right\}$$

Lemma 4

(a) $\mu^*$ is a well-defined outer measure on $X$ with $\mu^*(A) \leq \ell(1)$ for all $A \subset X$.

(b) $\mu^*(A) = \inf \left\{ \mu^*(O) \mid O \subset X, \, O \text{ open }, \, A \subset O \right\}$ for all $A \subset X$.

(c) If $O \subset X$ is open, then $O$ is measurable.

(d) If $U \subset X$ is Borel, then $U$ is measurable.

(e) If $A \subset X$ is measurable, then $\mu^*(A) = \sup \left\{ \mu^*(C) \mid C \subset A, \, C \text{ compact } \right\}$.

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Proof: (a) This follows almost directly from the definitions and the fact that, since \( \ell \) is a positive linear functional, \( 0 \leq \ell(f) \leq \ell(1) \) for any function \( f \in C(X) \) that obeys \( 0 \leq f(x) \leq 1 \) for all \( x \in X \).

(b) This is of course part of the definition of \( \mu^* \).

(c) Recall that, by definition, \( \mathcal{O} \subset X \) is measurable with respect to \( \mu^* \) if we have \( \mu^*(A) = \mu^*(A \cap \mathcal{O}) + \mu^*(A \cap (X \setminus \mathcal{O})) \) for all \( A \subset X \). So let \( A \subset X \). That \( \mu^*(A) \leq \mu^*(A \cap \mathcal{O}) + \mu^*(A \cap (X \setminus \mathcal{O})) \) is part of the definition of “outer measure”. So it suffices to prove that, for any \( \varepsilon > 0 \),

\[
\mu^*(A) \geq \mu^*(A \cap \mathcal{O}) + \mu^*(A \cap (X \setminus \mathcal{O})) - \varepsilon
\]

So fix any \( \varepsilon > 0 \). The definition of \( \mu^* \) is more direct when applied to open sets than to general sets, so we start by using the following argument to replace the general set \( A \) with an open set \( \tilde{A} \). By the definition of \( \mu^*(A) \), there is an open set \( \tilde{A} \subset X \) such that \( A \subset \tilde{A} \) and \( \mu^*(A) \geq \mu^*(\tilde{A}) - \frac{\varepsilon}{2} \). As \( A \subset \tilde{A} \), we have that \( \mu^*(\tilde{A} \cap \mathcal{O}) \geq \mu^*(A \cap \mathcal{O}) \) and \( \mu^*(\tilde{A} \cap (X \setminus \mathcal{O})) \geq \mu^*(A \cap (X \setminus \mathcal{O})) \). So it suffices to prove that

\[
\mu^*(\tilde{A}) \geq \mu^*(\tilde{A} \cap \mathcal{O}) + \mu^*(\tilde{A} \cap (X \setminus \mathcal{O})) - \frac{\varepsilon}{2}
\]

Here is the idea of the rest of the proof. We are going to construct three continuous functions \( f_1, f_2, f_3 : X \to [0, 1] \) and an open set \( \tilde{\mathcal{O}}^c \) that contains, but is only a tiny bit bigger than \( X \setminus \mathcal{O} \) (remember that \( \tilde{A} \cap (X \setminus \mathcal{O}) \) is not open), such that

\[
\begin{align*}
\mu^*(\tilde{A} \cap \mathcal{O}) &\leq \ell(f_1) + \frac{\varepsilon}{4} \quad f_1 \text{ nonzero only on } \tilde{A} \cap \mathcal{O} \\
\mu^*(\tilde{A} \cap (X \setminus \mathcal{O})) &\leq \ell(f_2) + \frac{\varepsilon}{4} \quad f_2 \text{ nonzero only on } \tilde{A} \cap \tilde{\mathcal{O}}^c \\
f_3 &= f_1 + f_2
\end{align*}
\]

Once we have succeeded in doing so, we have finished, since then \( f_3 \) is nonzero only on \( \tilde{A} \), takes values in \([0, 1]\) and is continuous so that

\[
\mu^*(\tilde{A} \cap \mathcal{O}) + \mu^*(\tilde{A} \cap (X \setminus \mathcal{O})) \leq \ell(f_1 + f_2) + \frac{\varepsilon}{2} = \ell(f_3) + \frac{\varepsilon}{2} \leq \mu^*(\tilde{A}) + \frac{\varepsilon}{2}
\]

So we now only have to construct \( f_1, f_2, f_3 \) and \( \tilde{\mathcal{O}}^c \). The principal hazard that we must avoid arises from the fact that \( \mathcal{O} \) and \( \tilde{\mathcal{O}}^c \) overlap a bit. So there is a danger that \( f_1 + f_2 \) is larger than \( 1 \) somewhere on \( \mathcal{O} \cap \tilde{\mathcal{O}}^c \). Fortunately, \( f_1 \) is zero on \( X \setminus \mathcal{O} \) and all of \( \mathcal{O} \cap \tilde{\mathcal{O}}^c \) is very close to \( X \setminus \mathcal{O} \), so \( f_1 \) is very small on \( \mathcal{O} \cap \tilde{\mathcal{O}}^c \). Here are the details.

Since \( \tilde{A} \cap \mathcal{O} \) is open, the definition of \( \mu^*(\tilde{A} \cap \mathcal{O}) \) implies that there is a continuous function \( F_1 : X \to [0, 1] \) that is nonzero only on \( \tilde{A} \cap \mathcal{O} \) and obeys \( \mu^*(\tilde{A} \cap \mathcal{O}) \leq \ell(F_1) + \frac{\varepsilon}{5} \). Since \( \ell(F_1) \leq \ell(1) < \infty \), we can pick a \( \delta > 0 \) such that \( \frac{\delta}{1+\delta} \ell(1) \leq \frac{\varepsilon}{20} \). Set \( f_1 = \frac{F_1}{1+\delta} \). Then

\[
\mu^*(\tilde{A} \cap \mathcal{O}) \leq \ell(F_1) + \frac{\varepsilon}{5} = \frac{1+\delta}{1+\delta} \ell(F_1) + \frac{\delta}{1+\delta} \ell(F_1) + \frac{\varepsilon}{5} \leq \ell(f_1) + \frac{\varepsilon}{4}
\]
Since \( F_1 \) is continuous and vanishes on \( X \setminus (\tilde{A} \cap \mathcal{O}) \), there is an open neighbourhood \( \tilde{O}^c \) of \( X \setminus (\tilde{A} \cap \mathcal{O}) \supset X \setminus \mathcal{O} \) such that \( F_1(x) \leq \delta \) for \( x \in \tilde{O}^c \). Since \( \tilde{A} \cap \tilde{O}^c \) is open, the definition of \( \mu^*(\tilde{A} \cap \tilde{O}^c) \) implies that there is a continuous function \( F_2 : X \to [0,1] \) that is nonzero only on \( \tilde{A} \cap \tilde{O}^c \) and obeys
\[
\mu^*(\tilde{A} \cap (X \setminus \mathcal{O})) \leq \mu^*(\tilde{A} \cap \tilde{O}^c) \leq \ell(F_2) + \frac{\xi}{5} = \frac{1}{1+\delta} \ell(F_2) + \frac{\delta}{1+\delta} \ell(F_2) + \frac{\xi}{5} \leq \frac{1}{1+\delta} \ell(F_2) + \frac{\xi}{4}
\]
Set \( f_2 = \frac{F_2}{1+\delta} \) and \( f_3 = f_1 + f_2 \). It remains only to verify that \( 0 \leq f_3(x) \leq 1 \), This follows from (see the figure below) the facts that
- \( f_1 \) is nonzero at most on \( \tilde{A} \cap \mathcal{O} \) and \( f_2 \) is nonzero at most on \( \tilde{A} \cap \tilde{O}^c \)
- on \( (\tilde{A} \cap \mathcal{O}) \cap \tilde{O}^c \), we have \( F_1 \leq \delta \) and \( F_2 \leq 1 \) so that \( f_3 = \frac{F_1}{1+\delta} + \frac{F_2}{1+\delta} \leq \frac{\delta}{1+\delta} + \frac{1}{1+\delta} = 1 \).
- on \( (\tilde{A} \cap \mathcal{O}) \setminus \tilde{O}^c \), we have \( f_1 \leq 1 \) and \( f_2 = 0 \)
- on \( \tilde{O}^c \setminus (\tilde{A} \cap \mathcal{O}) \), we have \( f_1 = 0 \) and \( f_2 \leq 1 \)

(d) By Carathéodory’s theorem, the set of all measurable sets is always a \( \sigma \)-algebra. In our case it contains all open sets and hence must contain all Borel sets.

(e) Let \( A \subset X \) be measurable. Then
\[
\mu^*(A) = \mu^*(X) - \mu^*(X \setminus A) = \mu^*(X) - \inf\{ \mu^*(\mathcal{O}) \mid \mathcal{O} \subset X, \, \mathcal{O} \text{ open }, \, X \setminus A \subset \mathcal{O} \} = \mu^*(X) - \inf\{ \mu^*(X) - \mu^*(X \setminus \mathcal{O}) \mid X \setminus \mathcal{O} \text{ compact }, \, X \setminus \mathcal{O} \subset A \} = \sup\{ \mu^*(C) \mid C \subset A, \, C \text{ compact } \}
\]

Proof of Theorem 3: Define \( \mu \) to be the restriction of \( \mu^* \) to the Borel sets. By Carathéodory’s theorem, \( \mu \) is a measure. By parts (b) and (d) of Lemma 4, it is a regular Borel measure. Since \( \mu(X) = \mu^*(X) = \ell(1) \), it is a finite measure. That \( \ell(f) = \int f(x) \, d\mu(x) \) is proven in Lemma 5, below.

That just leaves the uniqueness. If \( \nu \) is a regular Borel measure and \( \ell(f) = \int f(x) \, d\nu(x) \) for all \( f \in C(X) \), then we must have
\[
\nu^*(\mathcal{O}) = \sup\{ \ell(f) \mid f \in C(X), \, f \mid X \setminus \mathcal{O} = 0, \, 0 \leq f(x) \leq 1 \text{ for all } x \in X \} = \mu^*(\mathcal{O})
\]
for all open sets \( \mathcal{O} \). This was proven in the motivation leading up to the definition of \( \mu^* \). The regularity of \( \nu \) the forces \( \nu(A) = \mu^*(A) \) for all Borel sets \( A \).
Lemma 5 If $f \in C(X)$, then

$$\ell(f) = \int f(x) \, d\mu(x)$$

Proof: We first observe that it suffices to prove that $\ell(f) \leq \int f(x) \, d\mu(x)$ for all real-valued $f \in C(X)$. (Then $\ell(-f) \leq \int (-f)(x) \, d\mu$ too, so that $\ell(f) = \int f(x) \, d\mu(x)$ for all real-valued $f \in C(X)$.) We then observe that, since $\mu(X) = \ell(1) < \infty$, it suffices to consider $f \geq 0$. (Otherwise replace $f$ by $f + \|f\|_\infty$)

So fix any nonnegative $f \in C(X)$ and any $n \in \mathbb{N}$. Define, for each $m \in \mathbb{N}$,

$$B_m = f^{-1}\left(\left[\frac{m-1}{n}, \frac{m}{n}\right]\right)$$

This $B_m$ is the intersection of $f^{-1}\left(\left[\frac{m-1}{n}, \infty\right)\right)$, which is closed, and $f^{-1}\left(\left(-\infty, \frac{m}{n}\right]\right)$, which is open. So $B_m$ is Borel. Since $f$ is bounded, there is an $N \in \mathbb{N}$ such that $B_m = \emptyset$ for all $m > N$. For each $1 \leq m \leq N$, there is an open set $O_m \subset X$ such that $B_m \subset \mathcal{O}_m$, $\mu(B_m) \geq \mu(O_m) - \frac{1}{nN}$ and $0 \leq f \upharpoonright \mathcal{O}_m \leq \frac{m+1}{n}$, since $\mu$ is regular and $f$ is continuous. Again, for each $1 \leq m \leq N$, define

$$h_m(y) = \frac{d(y, X \setminus \mathcal{O}_m)}{\sum_{m'=1}^{N} d(y, X \setminus \mathcal{O}_{m'})}$$

and observe that

$$h_m \in C(X) \quad 0 \leq h_m \leq 1 \quad h_m(y) \neq 0 \iff y \in \mathcal{O}_m \quad \sum_{m=1}^{N} h_m(y) = 1$$

In particular, the denominator $\sum_{m'=1}^{N} d(y, X \setminus \mathcal{O}_{m'})$ never vanishes because each $y \in X$ is in $\mathcal{O}_{m'}$ for some $1 \leq m' \leq N$. (So $\{h_m\}_{1 \leq m \leq N}$ is a partition of unity. The only reason that it isn’t subordinate to the open cover $\{\mathcal{O}_m\}_{1 \leq m \leq N}$ is that the support of $h_m$ is $\overline{\mathcal{O}_{m'}}$.)

Hence

$$\ell(f) = \sum_{m=1}^{N} \ell(h_m f) \leq \sum_{m=1}^{N} \frac{m+1}{n} \ell(h_m) \leq \sum_{m=1}^{N} \frac{m+1}{n} \mu(\mathcal{O}_m) \leq \sum_{m=1}^{N} \frac{m+1}{n} \left[\mu(B_m) + \frac{1}{nN}\right]$$

$$= \sum_{m=1}^{N} \frac{m-1}{n} \mu(B_m) + \frac{2}{n} \sum_{m=1}^{N} \mu(B_m) + \sum_{m=1}^{N} \frac{m+1}{nN}$$

$$\leq \int f(x) \, d\mu(x) + \frac{2}{n} \mu(X) + \max_{1 \leq m \leq N} \frac{m+1}{n^2}$$

$$\leq \int f(x) \, d\mu(x) + \frac{2}{n} \mu(X) + \frac{1}{n}(\|f\|_\infty + \frac{1}{n})$$

For the first inequality we used the assumption that $\ell$ is positive. As (1) is true for all $n \in \mathbb{N}$, we have that $\ell(f) \leq \int f(x) \, \mu(x)$. □