Math 511 Problem Set 5
Due Thursday, November 15

Hand in solutions to 2, 4, 5, 6a, 7 only. You will be graded on clarity and readability as well as correctness. A mathematically correct answer will not earn full marks unless it is completely and rigorously justified, and written in a logical sequence that is easy to follow and confirm.

1. Let \( \{ e_n \}_{n \in \mathbb{N}} \) be an orthonormal basis for a Hilbert space \( \mathcal{H} \) and set \( e_\infty = \sum_{j=1}^{\infty} 2^{-j} e_j \). Define the linear operator \( T : D(T) \subset \mathcal{H} \rightarrow \mathbb{C} \) by

\[
D(T) = \left\{ \alpha e_\infty + \sum_{j=1}^{N} \beta_j e_j \mid N \in \mathbb{N}, \alpha, \beta_1, \ldots, \beta_N \in \mathbb{C} \right\} \quad T\left(\alpha e_\infty + \sum_{j=1}^{N} \beta_j e_j\right) = \alpha
\]

Prove that \( T \) is a well-defined linear operator but is not closable.

2. Give four examples of closed multiplication operators with
(a) one example having both domain and range closed and
(b) one example having domain closed but range not closed and
(c) one example having range closed but domain not closed and
(d) one example having both domain and range not closed.

3. Let \( \mathcal{H} \) be a Hilbert space and \( T : D(T) \subset \mathcal{H} \rightarrow \mathcal{H} \) be a densely defined linear operator. Prove that if \( \varphi \perp \text{range}(T) \), then \( \varphi \in D(T^*) \) and \( T^*\varphi = 0 \).

4. Let \( \mathcal{H} \) be a Hilbert space and \( D(A) \) be a dense linear subspace of \( \mathcal{H} \). Let \( A : D(A) \rightarrow \mathcal{H} \) be a symmetric linear operator. Let \( 0 \neq \mu_0 \in \mathbb{R} \). Prove that if the range \( R(A + i\mu_0 \mathbb{I}) = \mathcal{H} \), then \( R(A + i\mu \mathbb{I}) = \mathcal{H} \) for all nonzero \( \mu \in \mathbb{R} \) of the same sign as \( \mu_0 \).

5. Let \( \mathcal{H} = L^2(\mathbb{R}) \),

\[
D(A) = \left\{ f \in L^2(\mathbb{R}) \mid (1 + k^2)f(k) \in L^2(\mathbb{R}), \int f(k) \, dk = 0, \int kf(k) \, dk = 0 \right\}
\]

and \( A : D(A) \rightarrow L^2(\mathbb{R}) \) be defined by

\[
(Af)(k) = (1 + k^2)f(k)
\]

Find \( A^* \).

6. Let \( \mathcal{H} \) be a Hilbert space and \( A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H} \) be a closed linear operator. Let \( 0 \leq r < 1, R \in [0, \infty) \) and let \( B : D(B) \subset \mathcal{H} \rightarrow \mathcal{H} \) be another linear operator with \( D(A) \subset D(B) \) and

\[
\|B\varphi\| \leq r\|A\varphi\| + R\|\varphi\| \quad \text{for all } \varphi \in D(A)
\]

(a) Prove that \( A + B \), with domain \( D(A + B) = D(A) \), is again a closed operator.
(b) Prove that if \( A \) is self–adjoint and \( B \) is symmetric then, \( A + B \) with domain \( D(A + B) = D(A) \), is again self–adjoint.

\textit{Hint:} Show that if \( \mu > 0 \) is large enough, then \( A + B \pm i\mu \mathbb{I} = \{ \mathbb{I} + B(A \pm i\mu \mathbb{I})^{-1}\}(A \pm i\mu \mathbb{I}) \) has range \( \mathcal{H} \).

\textit{see over}
7. Let $D(T)$ be a dense subset of a Hilbert space $\mathcal{H}$ and let $T : D(T) \to \mathcal{H}$ be a closed linear operator. Two of the following statements are true. Prove them. One of the following statements is not true for at least one closed unbounded operator, though it is true for bounded operators. Exhibit a counterexample and say where the proof for the bounded case fails for the unbounded case.

(a) $\rho(T)$ is an open subset of $\mathbb{C}$
(b) $R_\lambda(T) = (\lambda I - T)^{-1}$ is an analytic function of $\lambda$ on $\rho(T)$.
(c) $\sigma(T) \neq \emptyset$.

see over