Math 511 Problem Set 1
Due Thursday, September 20

Hand in solutions to 2, 4, 5, 7, 8 only. You will be graded on clarity and readability as well as correctness. A mathematically correct answer will not earn full marks unless it is completely and rigorously justified, and written in a logical sequence that is easy to follow and confirm.

1. Let $\| \cdot \|$ be a norm on a vector space $V$. Prove that there exists an inner product $\langle \cdot, \cdot \rangle$ on $V$ such that

$$\langle x, x \rangle = \| x \|^2$$

for all $x \in V$ if and only if $\| \cdot \|$ obeys the parallelogram law $\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$ for all $x, y \in V$.

2. Let $D$ be an open subset of $\mathbb{C}$.
   (a) Let $z_0 \in D$. Prove that there is an $R > 0$, such that

$$\{ z \in \mathbb{C} \mid |z - z_0| \leq 3R \} \subset D,$$

and, if $f$ is an analytic function on $D$, then

$$f(z) = \frac{1}{2\pi R} \int_0^{2\pi} d\theta \int_{2R}^{3R} dr \, r e^{i\theta} \frac{f(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - z}$$

for all $z$ obeying $|z - z_0| \leq R$.
   (b) Let $\{ f_n \}_{n \in \mathbb{N}}$ be a Cauchy sequence in $A^2(D)$. Prove that there exists an analytic function $f$ on $D$ such that $\{ f_n \}_{n \in \mathbb{N}}$ converges uniformly to $f$ on compact subsets of $D$.

3. Let $\epsilon > 0$ and $-\infty < a < b < \infty$. Let $m$ be Lebesgue measure and $f : [a, b] \to \mathbb{R}$ be a Lebesgue–measurable function.
   (a) Prove that there exists an $M \in [0, \infty)$ such that

$$m\{ x \in [a, b] \mid |f(x)| \geq M \} \leq \epsilon$$

(b) Assume that $f : [a, b] \to [c, C]$. Prove that there exists a simple function $s$ such that $c \leq s(x) \leq C$ and $|f(x) - s(x)| \leq \epsilon$ for all $x \in [a, b]$. A simple function is, by definition, of the form $\sum_{j=1}^{n} a_j \chi_{E_j}(x)$ with $n \in \mathbb{N}$ and the sets $E_j$ measurable.
   (c) Let $s : [a, b] \to [c, C]$ be a simple function. Prove that there exists a step function $g : [a, b] \to [c, C]$ such that the measure

$$m\{ x \in [a, b] \mid s(x) \neq g(x) \} < \epsilon$$

A step function is, by definition, of the form $\sum_{i=1}^{n} a_i \chi_{E_i}(x)$ with $n \in \mathbb{N}$ and the sets $E_i$ intervals.
   (d) Let $g : [a, b] \to [c, C]$ be a step function. Prove that there exists a continuous function $h : [a, b] \to [c, C]$ such that $h(a) = h(b) = 0$ and

$$m\{ x \in [a, b] \mid g(x) \neq h(x) \} < \epsilon$$

see over
4. Let \(-\infty < a < b < \infty\) and \(1 \leq p < \infty\). Prove that the following sets of functions are dense in \(L^p([a,b])\).

(a) simple functions
(b) step functions
(c) continuous functions that vanish at \(a\) and \(b\)
(d) periodic \(C^\infty\) functions of period \(b-a\)
(e) \(C^\infty\) functions that are supported in \((a,b)\)

5. Prove that every vector space \(V \neq \{0\}\) has an algebraic basis.

*Hint:* Use Zorn’s Lemma (which is equivalent to the axiom of choice). It says that if a nonempty set \(S\)
(a) is partially ordered and
(b) has the property that every linearly ordered subset has an upper bound
then \(S\) has a maximal element.

6. Prove that \(\{e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in \mathbb{Z}}\) is an orthonormal basis for \(L^2([0,2\pi])\).

7. Let \(\mathcal{H}\) be an infinite dimensional Hilbert space. Construct a linear operator \(W : \mathcal{H} \rightarrow \mathcal{H}\) which is defined on all of \(\mathcal{H}\), but is not bounded. (Hint: use an algebraic basis.)

8. Let \((X,\mathcal{M},\mu)\) and \((Y,\mathcal{N},\nu)\) be \(\sigma\)-finite measure spaces and \(T : X \times Y \rightarrow \Phi\) be a function that is measurable with respect to \(\mathcal{M} \otimes \mathcal{N}\).

(a) Assume that
\[
M_1 = \text{ess sup}_{x \in X} \int_Y |T(x,y)| \, d\nu(y) < \infty
\]
\[
M_2 = \text{ess sup}_{y \in Y} \int_X |T(x,y)| \, d\mu(x) < \infty
\]
Let \(1 \leq p \leq \infty\). Prove that
\[
(T\varphi)(x) = \int_Y T(x,y)\varphi(y) \, d\nu(y)
\]
defines a bounded operator \(T : L^p(Y,\mathcal{N},\nu) \rightarrow L^p(X,\mathcal{M},\mu)\) with norm \(\|T\| \leq M_1^{\frac{1}{p}} M_2^{\frac{1}{q}}\).

(b) Assume that
\[
\|T\|_{\text{H.S.}} = \left[ \int_{X \times Y} |T(x,y)|^2 \, d\mu \times \nu (x,y) \right]^{1/2} < \infty
\]
Prove that \((T\varphi)(x) = \int_Y T(x,y)\varphi(y) \, d\nu(y)\) defines a bounded operator from \(L^2(Y,\mathcal{N},\nu)\) to \(L^2(X,\mathcal{M},\mu)\) with norm \(\|T\| \leq \|T\|_{\text{H.S.}}\).