Lattices and Periodic Functions

**Definition L.1** Let \( f(x) \) be a function on \( \mathbb{R}^d \).

a) The vector \( \gamma \in \mathbb{R}^d \) is said to be a period for \( f \) if

\[
f(x + \gamma) = f(x) \quad \text{for all } x \in \mathbb{R}^d
\]

b) Set

\[
\mathcal{P}_f = \{ \gamma \in \mathbb{R}^d \mid \gamma \text{ is a period for } f \}
\]

If \( \gamma, \gamma' \in \mathcal{P}_f \) then \( \gamma + \gamma' \in \mathcal{P}_f \) and if \( \gamma \in \mathcal{P}_f \) then \( -\gamma \in \mathcal{P}_f \) (sub \( x = z - \gamma \) into \( f(x + \gamma) = f(x) \)). Furthermore, the zero vector \( 0 \in \mathbb{R}^d \) is always in \( \mathcal{P}_f \). Thus \( \mathcal{P}_f \) is a (commutative) group under addition and

\[
\gamma_1, \ldots, \gamma_p \in \mathcal{P}_f \Rightarrow n_1 \gamma_1 + \cdots + n_p \gamma_p \in \mathcal{P}_f \quad \text{for all } p \in \mathbb{N} \text{ and } n_1, \ldots, n_p \in \mathbb{Z}
\]

**Example L.2**

a) If \( f(x, y) = \sin \left( \frac{2\pi x}{\ell_1} \right) \cos \left( \frac{2\pi y}{\ell_2} \right) \), then \( \mathcal{P}_f = \{ (m\ell_1, n\ell_2) \mid m, n \in \mathbb{Z} \} \).

b) If \( f(x, y) = \sin \left( \frac{2\pi x}{\ell_1} \right) \), then \( \mathcal{P}_f = \{ (m\ell_1, y) \mid m \in \mathbb{Z}, y \in \mathbb{R} \} \).

c) If \( f(x, y) = \sinh y \), then \( \mathcal{P}_f = \{ (m\ell_1, 0) \mid m \in \mathbb{Z} \} \).

To exclude functions, as in Example L.2.b, that are constant in some direction, it suffices to require that \( 0 \) be an isolated point of \( \mathcal{P}_f \). That is, to require that there be a number \( r > 0 \) such that every nonzero \( \gamma \in \mathcal{P}_f \) obeys \( |\gamma| \geq r \).

**Proposition L.3** If \( \mathcal{P} \) is an additive subgroup of \( \mathbb{R}^d \) and \( 0 \) is an isolated point of \( \mathcal{P} \), then there are \( d' \leq d \) and independent vectors \( \gamma_1, \ldots, \gamma_{d'} \in \mathbb{R}^d \) such that

\[
\mathcal{P} = \{ n_1 \gamma_1 + \cdots + n_{d'} \gamma_{d'} \mid n_1, \ldots, n_{d'} \in \mathbb{Z} \}
\]
Proof:

Claim 1. $\mathcal{P}$ has a shortest nonzero element.

Proof of Claim 1: Define $r = \inf \{ |\gamma| \mid \gamma \in \mathcal{P}, \gamma \neq 0 \}$. If there were no shortest element, there would be a sequence of vectors $\beta_1, \beta_2, \cdots$ in $\mathcal{P}$ with $\lim_{i \to \infty} |\beta_i| = r$ and $r < |\beta_i| \leq 2r$ for every $i = 1, 2, \cdots$. Because the closed ball of radius $2r$ is compact, the sequence has a limit point and hence has a Cauchy subsequence. In particular, there are $\beta_i$ and $\beta_j$ in the sequence, with $\beta_i \neq \beta_j$ with $|\beta_i - \beta_j| < \frac{r}{2}$. But this is impossible, because $\beta_i - \beta_j$ would be a nonzero element of $\mathcal{P}$ with length smaller than $r$.

Claim 2. Let $\gamma_1$ be a shortest nonzero element of $\mathcal{P}$ and set $\mathcal{P}_1 = \{ \gamma \in \mathcal{P} \mid \gamma \parallel \gamma_1 \}$. Then $\mathcal{P}_1 = \{ n\gamma_1 \mid n \in \mathbb{Z} \}$.

Proof of Claim 2: If $x\gamma_1 \in \mathcal{P}$ with $x$ not an integer, then $(x - [x])\gamma_1$ (where $[\cdot]$ denotes integer part) is a nonzero element of $\mathcal{P}$ with length strictly smaller than the length of $\gamma_1$. If $\mathcal{P} = \mathcal{P}_1$, we have finished. Otherwise continue with

Claim 3. Denote by $\mathbb{P}_1$ orthogonal projection in $\mathbb{R}^d$ onto the line $\{ x\gamma_1 \mid x \in \mathbb{R} \}$ and by $\mathbb{P}_1^\perp = \mathbb{1} - \mathbb{P}_1$ orthogonal projection perpendicular to the line $\{ x\gamma_1 \mid x \in \mathbb{R} \}$. Then $\mathcal{P} \setminus \mathcal{P}_1$ has an element whose distance from the line $\{ x\gamma_1 \mid x \in \mathbb{R} \}$ is a minimum, i.e. that minimizes $|\mathbb{P}_1^\perp \gamma|$.

Proof of Claim 3: Define $r_1 = \inf \{ |\mathbb{P}_1^\perp \gamma| \mid \gamma \in \mathcal{P} \setminus \mathcal{P}_1 \}$. If there were no minimizing element, there would be a sequence of vectors $\beta_1, \beta_2, \cdots$ in $\mathcal{P}$ with

$$2r_1 \geq |\mathbb{P}_1^\perp \beta_1| > |\mathbb{P}_1^\perp \beta_2| > |\mathbb{P}_1^\perp \beta_3| > \cdots > r_1$$

Because $|\mathbb{P}_1^\perp \beta| = |\mathbb{P}_1^\perp (\beta + n\gamma_1)|$ for all $n$, we may assume, without loss of generality, that $|\mathbb{P}_1 \beta| \leq |\gamma_1|$ for every $i$. Because

$$\{ x \in \mathbb{R}^d \mid |\mathbb{P}_1^\perp x| \leq 2r_1, |\mathbb{P}_1 x| \leq |\gamma_1| \}$$

is compact, the sequence has a limit point and hence has a Cauchy subsequence. In particular, there are $\beta_i$ and $\beta_j$ in the sequence, with $\beta_i \neq \beta_j$ with $|\beta_i - \beta_j| < \frac{r}{2}$. But this is impossible, because $\beta_i - \beta_j$ would be a nonzero element of $\mathcal{P}$ with length smaller than $r$.

Claim 4. Let $\gamma_2$ be an element of $\mathcal{P} \setminus \mathcal{P}_1$ that minimizes $|\mathbb{P}_1^\perp \gamma|$ and set

$$\mathcal{P}_2 = \mathcal{P} \cap \{ x_1 \gamma_1 + x_2 \gamma_2 \mid x_1, x_2 \in \mathbb{R} \}$$

Then $\mathcal{P}_2 = \{ n_1 \gamma_1 + n_2 \gamma_2 \mid n_1, n_2 \in \mathbb{Z} \}$.
Proof of Claim 4: If \( x_1 \gamma_1 + x_2 \gamma_2 \in \mathcal{P} \) with \( x_2 \) not an integer, then \( \gamma' = x_1 \gamma_1 + (x_2 - \lfloor x_2 \rfloor) \gamma_2 \) is an element of \( \mathcal{P} \setminus \mathcal{P}_1 \) with \( |\mathbb{P}_1 \gamma'| = |x_2 - \lfloor x_2 \rfloor| |\mathbb{P}_1 \gamma_2| < |\mathbb{P}_1 \gamma_2| \). So \( x_2 \) must be an integer. But then \( (x_1 \gamma_1 + x_2 \gamma_2) - x_2 \gamma_2 = x_1 \gamma_1 \in \mathcal{P} \) and, by Claim 2, \( x_1 \) must be an integer as well.

If \( \mathcal{P} = \mathcal{P}_2 \), we have finished. Otherwise continue with . . .

To exclude functions, as in Example L.2.c, that are "mixed periodic/non–periodic", we shall assume that \( d' = d \). Let \( \gamma_1, \ldots, \gamma_d \in \mathbb{R}^d \) be \( d \) linearly independent vectors and set

\[
\Gamma = \left\{ n_1 \gamma_1 + \cdots + n_d \gamma_d \mid n_1, \ldots, n_d \in \mathbb{Z} \right\}
\]

\( \Gamma \) is called the lattice generated by \( \gamma_1, \ldots, \gamma_d \).

**Problem L.1** The set of generators for a lattice are not uniquely determined. Let \( \Gamma \) be generated by \( d \) linearly independent vectors \( \gamma_1, \ldots, \gamma_d \in \mathbb{R}^d \). Let \( \Gamma' \) be generated by \( d \) linearly independent vectors \( \gamma_1', \ldots, \gamma_d' \in \mathbb{R}^d \). Prove that \( \Gamma = \Gamma' \) if and only there is a \( d \times d \) matrix \( A \) with integer matrix elements and \( |\det A| = 1 \) such that \( \gamma_i' = \sum_{j=1}^d A_{i,j} \gamma_j \).

**Problem L.2** Let \( \gamma_1, \ldots, \gamma_d \in \mathbb{R}^d \) be \( d \) linearly independent vectors. Prove that there are two constants \( C \) and \( c \), depending only on \( \gamma_1, \ldots, \gamma_d \) such that

\[
c |x| \leq |x_1 \gamma_1 + \cdots + x_d \gamma_d| \leq C |x|
\]

for all \( x \in \mathbb{R}^d \).

We’ll now find a bunch of functions that are periodic with respect to \( \Gamma \). Consider \( f(x) = e^{ib \cdot x} \). This function has period \( \gamma \) if and only if \( e^{ib \cdot (x+\gamma)} = e^{ib \cdot x} \) for all \( x \in \mathbb{R}^d \). This is the case if and only if \( e^{ib \cdot \gamma} = 1 \) and this is the case if and only if \( b \cdot \gamma \in 2\pi \mathbb{Z} \).

**Definition L.4** Let \( \Gamma \) be a lattice in \( \mathbb{R}^d \). The dual lattice for \( \Gamma \) is

\[
\Gamma^\# = \left\{ b \in \mathbb{R}^d \mid b \cdot \gamma \in 2\pi \mathbb{Z} \text{ for all } \gamma \in \Gamma \right\}
\]

**Remark L.5** Let \( \gamma_1, \ldots, \gamma_d \in \mathbb{R}^d \) be linearly independent and denote by \( \Gamma \) the lattice that they generate. A vector \( b \in \mathbb{R}^d \) is an element of \( \Gamma^\# \) if and only if

\[
b \cdot \gamma_j \in 2\pi \mathbb{Z} \quad \text{for all } 1 \leq j \leq d
\]
Example L.6 Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. That is, $e_j$ has all components zero, except for the $j$th, which is one. Choosing $\ell_1, \ldots, \ell_d > 0$ and $\gamma_j = \ell_j e_j$,

$$\Gamma = \{ (n_1 \ell_1, \ldots, n_d \ell_d) \mid n_1, \ldots, n_d \in \mathbb{Z} \}$$

Then $(x_1, \ldots, x_d)$ is in $\Gamma^\#$ if and only if

$$(x_1, \ldots, x_d) \cdot \gamma_j = \ell_j x_j \in 2\pi \mathbb{Z} \iff x_j \in \frac{2\pi}{\ell_j} \mathbb{Z}$$

Thus

$$\Gamma^\# = \{ (n_1 \frac{2\pi}{\ell_1}, \ldots, n_d \frac{2\pi}{\ell_d}) \mid n_1, \ldots, n_d \in \mathbb{Z} \}$$

Example L.7 Let

$$\Gamma = \{ n(1,0) + m(\pi,1) \mid n, m \in \mathbb{Z} \}$$

Then

$$\Gamma^\# = \{ n(0,2\pi) + m(2\pi,-2\pi^2) \mid n, m \in \mathbb{Z} \}$$

Since

$$\left[n'(1,0) + m'\pi,1\right] \cdot \left[n(0,2\pi) + m(2\pi,-2\pi^2)\right] = 2\pi (n'm + m'n)$$

every vector of the form $n(0,2\pi) + m(2\pi,-2\pi^2)$ with $m, n \in \mathbb{Z}$ is indeed in $\Gamma^\#$. To verify that only vectors of this form are in $\Gamma^\#$, let $z = x(0,2\pi) + y(2\pi,-2\pi^2)$ be any vector in $\mathbb{R}^2$. (Certainly, $(0,2\pi)$ and $(2\pi,-2\pi^2)$ form a basis for $\mathbb{R}^2$.) For $z$ to be in $\Gamma^\#$ it is necessary that

$$z \cdot (1,0) = 2\pi y \in 2\pi \mathbb{Z}$$

$$z \cdot (\pi,1) = 2\pi x \in 2\pi \mathbb{Z}$$

which forces $x, y \in \mathbb{Z}$.

Problem L.3 Let $\Gamma$ be generated by $\gamma_1, \ldots, \gamma_d \in \mathbb{R}^d$ (assumed linearly independent) and let

$$[\gamma_1, \ldots, \gamma_d] = \{ \sum_{j=1}^d t_j \gamma_j \mid 0 \leq t_j \leq 1 \text{ for all } 1 \leq j \leq d \}$$

be the parallelepiped with the $\gamma_j$’s as edges. Prove that if $b \in \Gamma^\#$, then

$$\int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{i b \cdot x} = \begin{cases} ||[\gamma_1, \ldots, \gamma_d]|| & \text{if } b = 0 \\ 0 & \text{if } b \neq 0 \end{cases}$$

where $|[\gamma_1, \ldots, \gamma_d]|$ is the volume of $|[\gamma_1, \ldots, \gamma_d]|$. By Problem L.1, the volume $|[\gamma_1, \ldots, \gamma_d]|$ is independent of the choice of generators. That is, if $\Gamma$ is also generated by $\gamma'_1, \ldots, \gamma'_d \in \mathbb{R}^d$, then $|[\gamma_1, \ldots, \gamma_d]| = |[\gamma'_1, \ldots, \gamma'_d]|$. Consequently, it is legitimate to define $|\Gamma| = |[\gamma_1, \ldots, \gamma_d]|$. Hence

$$\frac{1}{|\Gamma|} \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{i b \cdot x} = \begin{cases} 1 & \text{if } b = 0 \\ 0 & \text{if } b \neq 0 \end{cases}$$
Proposition L.8 If $\gamma_1, \ldots, \gamma_d \in \mathbb{R}^d$ are linearly independent and
\[
\Gamma = \{ n_1\gamma_1 + \cdots + n_d\gamma_d \mid n_1, \ldots, n_d \in \mathbb{Z} \}
\]
then there exist $d$ linearly independent vectors $b_1, \ldots, b_d \in \mathbb{R}^d$ such that
\[
\Gamma^\# = \{ n_1b_1 + \cdots + n_db_d \mid n_1, \ldots, n_d \in \mathbb{Z} \}
\]

Proof: For each $1 \leq i \leq d$ \[
V_i = \{ x_1\gamma_1 + \cdots + x_d\gamma_d \mid x_1, \ldots, x_d \in \mathbb{R}, x_i = 0 \}
\]
is a $d-1$ dimensional subspace of $\mathbb{R}^d$. So $V_i^\perp$ is a one dimensional subspace of $\mathbb{R}^d$. Let $B_i$ be any nonzero element of $V_i^\perp$ and define \[
b_i = \frac{2\pi}{\gamma_i \cdot B_i} B_i
\]
Note that $\gamma_i \cdot B_i$ cannot vanish because then $\gamma_i$ would have to be in $V_i$, i.e. would have to be a linear combination of $\gamma_j$, $j \neq i$. Denote \[
B = \{ n_1b_1 + \cdots + n_db_d \mid n_1, \ldots, n_d \in \mathbb{Z} \}
\]
As \[
b_i \cdot \gamma_j = \begin{cases} 2\pi & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]
we have that $b_i \in \Gamma^\#$ and hence $B \subset \Gamma^\#$.

If $x_1b_1 + \cdots + x_db_d = 0$ then $(x_1b_1 + \cdots + x_db_d) \cdot \gamma_j = 2\pi x_j = 0$ for every $1 \leq j \leq d$. So the $b_i$'s are linearly independent and every vector in $\mathbb{R}^d$ may be written in the form $x_1b_1 + \cdots + x_db_d$. If $x_1b_1 + \cdots + x_db_d \in \Gamma^\#$, then \[
(x_1b_1 + \cdots + x_db_d) \cdot \gamma_j = 2\pi x_j \in 2\pi \mathbb{Z}
\]
so that $x_j \in \mathbb{Z}$ for every $1 \leq j \leq d$. Hence $\gamma^\# \subset B$. \qed

From now on, we fix $d$ linearly independent vectors $\gamma_1, \ldots, \gamma_d \in \mathbb{R}^d$, set \[
\Gamma = \{ n_1\gamma_1 + \cdots + n_d\gamma_d \mid n_1, \ldots, n_d \in \mathbb{Z} \}
\]
The set of all $C^\infty$ functions on $\mathbb{R}^d$ that are periodic with respect to $\Gamma$ is denoted $C^\infty(\mathbb{R}^d/\Gamma)$. We have already observed that $f(x) = e^{ib \cdot x}$ is in $C^\infty(\mathbb{R}^d/\Gamma)$ if and only in $b \in \Gamma^\#$.
**Remark L.9** Here is the story (at least in short form) behind the notation $C^\infty(\mathbb{R}^d/\Gamma)$. We have already observed that $\mathbb{R}^d$ is a group (under addition) and that $\Gamma$ is a subgroup of $\mathbb{R}^d$. As $\mathbb{R}^d$ is abelian, all subgroups are normal and the set of equivalence classes under the equivalence relation

$$x \sim y \iff x - y \in \Gamma$$

is itself a group, denoted, as usual $\mathbb{R}^d/\Gamma$. Precisely, the equivalence class of $x \in \mathbb{R}^d$ is $[x] = \{ y \in \mathbb{R}^d \mid x \sim y \} \subset \mathbb{R}^d$ and $\mathbb{R}^d/\Gamma = \{ [x] \mid x \in \mathbb{R}^d \}$. The group operation in $\mathbb{R}^d/\Gamma$ is $[x] + [y] = [x + y]$.

As well as being a group, $\mathbb{R}^d/\Gamma$ can also be turned into a smooth manifold, called a $d$–dimensional torus. If $\mathcal{O}$ is any open subset of $\mathbb{R}^d$ with the property that no two points of $\mathcal{O}$ are equivalent under $\sim$, then the map

$$\xi_{\mathcal{O}} : \mathcal{O} \to \mathbb{R}^d/\Gamma$$

$$x \mapsto [x]$$

is one–to–one. Its inverse is a coordinate map for $\mathbb{R}^d/\Gamma$. If $\mathcal{O}$ is generated by $\gamma_1, \ldots, \gamma_d$ and $X$ is any point in $\mathbb{R}^d$, $\{ X + t_1\gamma_1 + \cdots + t_d\gamma_d \mid 0 < t_j < 1 \text{ for all } 1 \leq j \leq d \}$ is one possible choice of $\mathcal{O}$. The notation $C^\infty(\mathbb{R}^d/\Gamma)$ designates the set of smooth (that is, $C^\infty$) functions on the manifold $\mathbb{R}^d/\Gamma$.

**Theorem L.10 (Fourier Series)** A function $f : \mathbb{R}^d \to \mathbb{C}$ is in $C^\infty(\mathbb{R}^d/\Gamma)$ if and only if

$$f(x) = \frac{1}{|\Gamma|} \sum_{b \in \Gamma^\#} \hat{f}_b e^{ib \cdot x}$$

with $\sum_{b \in \Gamma^\#} |b|^{2n} \hat{f}_b < \infty$ for all $n \in \mathbb{N}$

Furthermore, in this case,

$$\hat{f}_b = \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{-ib \cdot x} f(x)$$

**Proof of “if”:** Suppose that we are given $\hat{f}_b$, $b \in \Gamma^\#$ obeying $\sum_{b \in \Gamma^\#} |b|^{2n} \hat{f}_b < \infty$ for all $n \in \mathbb{N}$. In particular $\sum_{b \in \Gamma^\#} |\hat{f}_b| < \infty$ so the series $\frac{1}{|\Gamma|} \sum_{b \in \Gamma^\#} \hat{f}_b e^{ib \cdot x}$ converges absolutely and uniformly to some continuous function that is periodic with respect to $\Gamma$. Call the function $f(x)$. Furthermore for any $i_1, \ldots, i_d \in \mathbb{N}$

$$\left| \left( \prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j} \right) \hat{f}_b e^{ib \cdot x} \right| = \left| \left( \prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_b e^{ib \cdot x} \right| \leq |b|^{\Sigma_i} \hat{f}_b$$
so the series \( \frac{1}{|\Gamma|} \sum_{b \in \Gamma^*} \left( \prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} \right) \hat{f}_b e^{ib \cdot x} \) also converges absolutely and uniformly. This implies that \( f(x) \) is \( C^\infty \). Furthermore

\[
\int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{-ib \cdot x} f(x) = \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{-ib \cdot x} \left[ \frac{1}{|\Gamma|} \sum_{c \in \Gamma^*} \hat{f}_c e^{ic \cdot x} \right]
= \frac{1}{|\Gamma|} \sum_{c \in \Gamma^*} \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{i(c-b) \cdot x} \hat{f}_c
= \frac{1}{|\Gamma|} \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ \hat{f}_b + \frac{1}{|\Gamma|} \sum_{c \in \Gamma^* \setminus b} \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{i(c-b) \cdot x} \hat{f}_c
= \hat{f}_b
\]

by Problem L.3.

**Proof of “only if”:** Now suppose that we are given \( f \in C^\infty \left( \mathbb{R}^d / \Gamma \right) \). Define

\[
\hat{f}_b = \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{-ib \cdot x} f(x)
\]

Then for any \( i_1, \ldots, i_d \in \mathbb{N} \)

\[
\left| \left( \prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_b \right| = \left| \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ \left( \prod_{j=1}^d b_{i_j}^{i_j} \right) e^{-ib \cdot x} f(x) \right|
= \left| \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ \left( \prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} e^{-ib \cdot x} \right) f(x) \right|
= \left| \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{-ib \cdot x} \left( \prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} f(x) \right) \right|
\leq |\Gamma| \sup_x \left( \prod_{j=1}^d \frac{\partial^{i_j}}{\partial x_j^{i_j}} f(x) \right) < \infty
\]

so that, by Problem L.2,

\[
\sum_{b \in \Gamma^*} \left| \left( \prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_b \right| = \sum_{b \in \Gamma^*} \frac{1 + |b|^{d+1}}{1 + |b|^{a+d+1}} \left| \left( \prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_b \right|
\leq \left[ \sup_{b \in \Gamma^*} \left( 1 + |b|^{d+1} \right) \left| \left( \prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_b \right| \right] \sum_{b \in \Gamma^*} \frac{1}{1 + |b|^{a+d+1}}
\leq \left[ \sup_{b \in \Gamma^*} \left( 1 + |b|^{d+1} \right) \left| \left( \prod_{j=1}^d b_{i_j}^{i_j} \right) \hat{f}_b \right| \right] \sum_{n \in \mathbb{Z}^d} \frac{1}{1 + (c|n|)^{d+1}} < \infty
\]
Hence, by the “only if” part of this Theorem, that we have already proven,

\[ g(x) = \frac{1}{|\Gamma|} \sum_{b \in \Gamma^\#} \hat{f}_b e^{ib \cdot x} \]

is a \( C^\infty \) function and

\[ \hat{f}_b = \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{-ib \cdot x} g(x) \] \hspace{1cm} (L.1)

We just have to show that \( g(x) = f(x) \).

Here is one proof that \( g(x) = f(x) \). By (L.1)

\[ \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ e^{-ib \cdot x} [g(x) - f(x)] = 0 \]

for all \( b \in \Gamma^\# \). Consequently,

\[ \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ \varphi(x) [g(x) - f(x)] = 0 \]

for any function \( \varphi \in \mathcal{P}(\Gamma^\#) \) where \( \mathcal{P}(\Gamma^\#) \) is the set of all functions that are finite linear combinations of the \( e^{-ib \cdot x} \)'s with \( b \in \Gamma^\# \). Consequently,

\[ \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ \varphi(x) [g(x) - f(x)] = 0 \]

for any function \( \varphi \in \overline{\mathcal{P}(\Gamma^\#)} \) where \( \overline{\mathcal{P}(\Gamma^\#)} \) is the set of all functions that are uniform limits of sequences of functions in \( \mathcal{P}(\Gamma^\#) \). But by the Stone–Weierstrass Theorem [Walter Rudin, Principles of Mathematical Analysis, Theorem 7.33], \( \overline{\mathcal{P}(\Gamma^\#)} \) is the set of all continuous functions that are periodic with respect to \( \Gamma \). In particular, the complex conjugate of \( g(x) - f(x) \) is in \( \overline{\mathcal{P}(\Gamma^\#)} \). Hence

\[ \int_{[\gamma_1, \ldots, \gamma_d]} d^d x \ |g(x) - f(x)|^2 = 0 \]

so that \( g(x) = f(x) \) for all \( x \).

One may also build Problem L.5, below, into a second proof that \( g(x) = f(x) \). Just make a change of variables so that \( \Gamma \) is replaced by \( 2\pi \mathbb{Z}^d \) and apply Problem L.5.b, once in each dimension.

**Problem L.4** Let \( \Gamma \) be generated by \( \gamma_1, \ldots, \gamma_d \in \mathbb{R}^d \) (assumed linearly independent) and also by \( \gamma'_1, \ldots, \gamma'_d \in \mathbb{R}^d \) (also assumed linearly independent). Recall that

\[ [\gamma_1, \ldots, \gamma_d] = \left\{ \sum_{j=1}^d t_j \gamma_j \mid 0 \leq t_j \leq 1 \text{ for all } 1 \leq j \leq d \right\} \]
is the parallelepiped with the $\gamma_j$’s as edges. Let, for $y \in \mathbb{R}^d$,

$$y + [\gamma_1, \cdots, \gamma_d] = \{ y + \sum_{j=1}^{d} t_j \gamma_j \mid 0 \leq t_j \leq 1 \text{ for all } 1 \leq j \leq d \}$$

be the translate of $[\gamma_1, \cdots, \gamma_d]$ by $y$. Let $f(x)$ be periodic with respect to $\Gamma$. Prove that

$$\int_{[\gamma_1, \cdots, \gamma_d]} d^d x \ f(x) = \int_{y + [\gamma_1, \cdots, \gamma_d]} d^d x \ f(x) = \int_{[\gamma_1', \cdots, \gamma_d']} d^d x \ f(x)$$

We denote

$$\int_{\mathbb{R}^d / \Gamma} d^d x \ f(x) = \int_{[\gamma_1, \cdots, \gamma_d]} d^d x \ f(x)$$

**Problem L.5** Let $f \in C^1(\mathbb{R})$ be periodic of period $2\pi$. Set

$$c_n = \int_{0}^{2\pi} e^{-inx} f(x) \ dx$$

and

$$(S_M f)(\theta) = \frac{1}{2\pi} \sum_{n=-M}^{M} c_n e^{in\theta}$$

a) Prove that $S_M f(\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta + x) \frac{\sin(M+1/2)x}{\sin(x/2)} \ dx$.

b) Prove that $S_M f(\theta)$ converges to $f(\theta)$ as $M \to \infty$. 