Families of Commuting Normal Matrices

Definition M.1 (Commuting) Two \( n \times n \) matrices \( A \) and \( B \) are said to commute if \( AB = BA \).

Definition M.2 (Adjoint) The adjoint of the \( r \times c \) matrix \( A \) is the \( c \times r \) matrix

\[
A_{i,j}^* = \overline{A_{j,i}}
\]

Here \( \overline{A_{j,i}} \) is the complex conjugate of \( A_{j,i} \).

Problem M.1 Let \( A \) and \( B \) be any \( n \times n \) matrices. Prove that \( B = A^* \) if and only if \( \langle Bv, w \rangle = \langle v, Aw \rangle \) for all \( v, w \in \mathbb{C}^n \). Here \( \langle v, w \rangle = \sum_{j=1}^{n} \overline{v_j}w_j \) is the inner product of the vectors \( v = (v_1, \ldots, v_n) \) and \( w = (w_1, \ldots, w_n) \).

Problem M.2 Let \( A \) be any \( n \times n \) matrix. Let \( V \) be any linear subspace of \( \mathbb{C}^n \) and \( V^\perp \) its orthogonal complement. Prove that if \( AV \subset V \) (i.e. \( w \in V \Rightarrow Aw \in V \)), then \( A^*V^\perp \subset V^\perp \).

Definition M.3 (Normal, Self–Adjoint, Unitary)

i) An \( n \times n \) matrix \( A \) is normal if \( AA^* = A^*A \). That is, if \( A \) commutes with its adjoint.

ii) An \( n \times n \) matrix \( A \) is self–adjoint if \( A = A^* \).

iii) An \( n \times n \) matrix \( U \) is unitary if \( U^*U = \mathbb{I} \). Here \( \mathbb{I} \) is the \( n \times n \) identity matrix. Its \( (i,j) \) matrix element is one if \( i = j \) and zero otherwise.

Problem M.3 Let \( A \) be a normal matrix. Let \( \lambda \) be an eigenvalue of \( A \) and \( V \) the eigenspace of \( A \) of eigenvalue \( \lambda \). Prove that \( V \) is the eigenspace of \( A^* \) of eigenvalue \( \overline{\lambda} \).

Problem M.4 Let \( A \) be a normal matrix. Let \( v \) and \( w \) be eigenvectors of \( A \) with different eigenvalues. Prove that \( v \perp w \).

Problem M.5 Let \( A \) be a self-adjoint matrix. Prove that

a) \( A \) is normal

b) Every eigenvalue of \( A \) is real.
Problem M.6 Let $U$ be a unitary matrix. Prove that

a) $U$ is normal

b) Every eigenvalue $\lambda$ of $U$ obeys $|\lambda| = 1$, i.e. is of modulus one.

The main result in these notes is

Theorem M.4 Let $n \geq 1$ be an integer. Let $\mathcal{F}$ be a nonempty set of $n \times n$ mutually commuting normal matrices. That is, $A, B \in \mathcal{F} \Rightarrow AB = BA$ and $A \in \mathcal{F} \Rightarrow AA^* = A^*A$. Then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ such that $e_j$ is an eigenvector of $A$ for every $A \in \mathcal{F}$ and every $1 \leq j \leq n$.

The proof uses two lemmas.

Lemma M.5 Let $V$ be a linear subspace of $\mathbb{C}^n$ of dimension at least one. Let $A$ be an $n \times n$ matrix that maps $V$ into $V$. Then $A$ has an eigenvector in $V$.

Proof: Let $e_1, \ldots, e_d$ be a basis for $V$. As $A$ maps $V$ into itself, there exist numbers $a_{i,j}$, $1 \leq i, j \leq d$ such that

$$Ae_j = \sum_{i=1}^{d} a_{i,j} e_i \quad \text{for all } 1 \leq j \leq d$$

Consequently, $A$ maps the vector $w = \sum_{j=1}^{d} x_j e_j \in V$ to

$$Aw = \sum_{i,j=1}^{d} a_{i,j} x_j e_i$$

so that $w$ is an eigenvector of $A$ of eigenvalue $\lambda$ if and only if (1) not all of the $x_i$’s are zero and (2)

$$\sum_{i,j=1}^{d} a_{i,j} x_j e_i = \lambda \sum_{i=1}^{d} x_i e_i \iff \sum_{j=1}^{d} a_{i,j} x_j = \lambda x_i \quad \text{for all } 1 \leq i \leq d$$

$$\iff \sum_{j=1}^{d} (a_{i,j} - \lambda \delta_{i,j}) x_j = 0 \quad \text{for all } 1 \leq i \leq d$$

For any given $\lambda$, the linear system of equations “$\sum_{j=1}^{d} (a_{i,j} - \lambda \delta_{i,j}) x_j = 0$ for all $1 \leq i \leq d$” has a nontrivial solution $(x_1, \ldots, x_d)$ if and only if the $d \times d$ matrix $[a_{i,j} - \lambda \delta_{i,j}]_{1 \leq i, j \leq d}$ fails to be invertible and this, in turn, is the case if and only if $\det [a_{i,j} - \lambda \delta_{i,j}] = 0$. But $\det [a_{i,j} - \lambda \delta_{i,j}] = 0$ is a polynomial of degree $d$ in $\lambda$ and so always vanishes for at least one value of $\lambda$. ■
**Lemma M.6** Let \( n \geq 1 \) be an integer, \( V \) be a linear subspace of \( \mathbb{C}^n \) of dimension at least one and let \( \mathcal{F} \) be a nonempty set of \( n \times n \) mutually commuting matrices that map \( V \) into \( V \). That is, \( A, B \in \mathcal{F} \Rightarrow AB = BA \) and \( A \in \mathcal{F}, \ w \in V \Rightarrow Aw \in V \). Then there exists a nonzero vector \( v \in V \) that is an eigenvector for every matrix in \( \mathcal{F} \).

**Proof:** We shall show that

“There is a linear subspace \( W \) of \( V \) of dimension at least one, such that each \( A \in \mathcal{F} \) is a multiple of the identity matrix when restricted to \( W \).”

This suffices to prove the lemma. The proof will be by induction on the dimension \( d \) of \( V \). If \( d = 1 \), we may take \( W = V \), since the restriction of any matrix to a one dimensional vector space is a multiple of the identity.

Suppose that the claim has been proven for all dimensions strictly less than \( d \). If every \( A \in \mathcal{F} \) is a multiple of the identity, when restricted to \( V \), we may take \( W = V \) and we are done. If not, pick any \( A \in \mathcal{F} \) that is not a multiple of the identity when restricted to \( V \). By Lemma M.5, it has at least one eigenvector \( v \in V \). Let \( \lambda \) be the corresponding eigenvalue and set

\[
V' = V \cap \{ \ w \in \mathbb{C}^n \mid Aw = \lambda w \} \]

Then \( V' \) is a linear subspace of \( V \) of dimension strictly less than \( d \) (since \( A \), restricted to \( V \), is not \( \lambda I \)). We claim that every \( B \in \mathcal{F} \) maps \( V' \) into \( V' \). To see this, let \( B \in \mathcal{F} \) and \( w \in V' \) and set \( w' = Bw \). We wish to show that \( w' \in V' \). But

\[
Aw' = ABw = BAw \quad (A \text{ and } B \text{ commute})
\]

\[
= B\lambda w \quad \text{(Definition of } V')
\]

\[
= \lambda Bw = \lambda w'
\]

so \( w' \) is indeed in \( V' \). We have verified that \( V' \) has dimension at least one and strictly smaller than \( d \) and that every \( B \in \mathcal{F} \) maps \( V' \) into \( V' \). So we may apply the inductive hypothesis with \( V \) replaced by \( V' \).

**Proof of Theorem M.4:** By Lemma M.6, with \( V = \mathbb{C}^n \), there exists a nonzero vector \( v_1 \) that is an eigenvector for every \( A \in \mathcal{F} \). Set \( e_1 = \frac{v_1}{\|v_1\|} \) and \( V_1 = \{ \lambda e_1 \mid \lambda \in \mathbb{C} \} \). By Problem M.3, \( e_1 \) is also an eigenvector of \( A^* \) for every \( A \in \mathcal{F} \), so \( A^*V_1 \subset V_1 \) for all \( A \in \mathcal{F} \). By Problem M.2, \( AV_1^\perp \subset V_1^\perp \) for all \( A \in \mathcal{F} \).

By Lemma M.6, with \( V = V_1^\perp \), there exists a nonzero vector \( v_2 \in V_1^\perp \) that is an eigenvector for every \( A \in \mathcal{F} \). Choose \( e_2 = \frac{v_2}{\|v_2\|} \). As \( e_2 \in V_1^\perp \), \( e_2 \) is orthogonal to \( e_1 \). Define \( V_2 = \{ \lambda_1 e_1 + \lambda_2 e_2 \mid \lambda_1, \lambda_2 \in \mathbb{C} \} \). By Problem M.3, \( e_2 \) is also an eigenvector of \( A^* \) for every \( A \in \mathcal{F} \), so \( A^*V_2 \subset V_2 \) for all \( A \in \mathcal{F} \). By Problem M.2, \( AV_2^\perp \subset V_2^\perp \) for all \( A \in \mathcal{F} \).
By Lemma M.6, with $V = V_2^\perp$, there exists a nonzero vector $v_3 \in V_2^\perp$ that is an eigenvector for every $A \in \mathcal{F}$. Choose $e_3 = \frac{v_3}{\|v_3\|}$. As $e_3 \in V_2^\perp$, $e_3$ is orthogonal to both $e_1$ and $e_2$. And so on. ■