Review of Hilbert and Banach Spaces

Definition 1 (Vector Space) A vector space over $\mathbb{C}$ is a set $\mathcal{V}$ equipped with two operations,

$$(\mathbf{v}, \mathbf{w}) \in \mathcal{V} \times \mathcal{V} \mapsto \mathbf{v} + \mathbf{w} \in \mathcal{V} \qquad\qquad (\alpha, \mathbf{v}) \in \mathbb{C} \times \mathcal{V} \mapsto \alpha\mathbf{v} \in \mathcal{V}$$

called addition and scalar multiplication, respectively, that obey the following axioms.

Additive Axioms. There is an element $\mathbf{0} \in \mathcal{V}$ and, for each $\mathbf{x} \in \mathcal{V}$ there is an element $-\mathbf{x} \in \mathcal{V}$ such that, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$,

(i) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

(ii) $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$

(iii) $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$

(iv) $(-\mathbf{x}) + \mathbf{x} = \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

Multiplicative Axioms. For every $\mathbf{x} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$,

(v) $0\mathbf{x} = \mathbf{0}$

(vi) $1\mathbf{x} = \mathbf{x}$

(vii) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$

Distributive Axioms. For every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$,

(viii) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$

(ix) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$

Definition 2 (Subspace) A subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is called a *linear subspace* of $\mathcal{V}$ if it is closed under addition and scalar multiplication. That is, if $\mathbf{x} + \mathbf{y} \in \mathcal{W}$ and $\alpha\mathbf{x} \in \mathcal{W}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{W}$ and all $\alpha \in \mathbb{C}$. Then $\mathcal{W}$ is itself a vector space over $\mathbb{C}$.

Definition 3 (Inner Product)

(a) A inner product on a vector space $\mathcal{V}$ is a function $(\mathbf{x}, \mathbf{y}) \in \mathcal{V} \times \mathcal{V} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{C}$ that obeys

(i) (Linearity in the second argument) $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$, $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$

(ii) (Conjugate symmetry) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$

(iii) (Positive–definiteness) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ if $\mathbf{x} \neq \mathbf{0}$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha \in \mathbb{C}$.

(b) Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal* with respect to the inner product $\langle \cdot, \cdot \rangle$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

(c) We'll use the terms "inner product space" or "pre–Hilbert space" to mean a vector space over $\mathbb{C}$ equipped with an inner product.

Definition 4 (Norm)

(a) A norm on a vector space $\mathcal{V}$ is a function $\mathbf{x} \in \mathcal{V} \mapsto \|\mathbf{x}\| \in [0, \infty)$ that obeys

(i) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(ii) $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$

(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha \in \mathbb{C}$.

(b) A sequence $\{\mathbf{v}_n\}_{n \in \mathbb{N}} \subset \mathcal{V}$ is said to be *Cauchy* with respect to the norm $\| \cdot \|$ if

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \text{s.t.} \;\; m, n > N \implies \|\mathbf{v}_n - \mathbf{v}_m\| < \varepsilon$$

(c) A sequence $\{\mathbf{v}_n\}_{n \in \mathbb{N}} \subset \mathcal{V}$ is said to *converge* to $\mathbf{v}$ in the norm $\| \cdot \|$ if

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \text{s.t.} \;\; n > N \implies \|\mathbf{v}_n - \mathbf{v}\| < \varepsilon$$

(d) A normed vector space is said to be *complete* if every Cauchy sequence converges.

(e) A subset $\mathcal{D}$ of a normed vector space $\mathcal{V}$ is said to be *dense* in $\mathcal{V}$ if $\overline{\mathcal{D}} = \mathcal{V}$, where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$. That is, if every element of $\mathcal{V}$ is a limit of a sequence of elements of $\mathcal{D}$.

Theorem 5 *Let $\langle \cdot \, , \, \cdot \rangle$ be an inner product on a vector space $\mathcal{V}$ and set $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ for all $\mathbf{x} \in \mathcal{V}$.*

(a) The inner product is sesquilinear. That is,

$$\langle \mathbf{x}, \alpha \mathbf{y} + \beta \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle + \beta \langle \mathbf{x}, \mathbf{z} \rangle$$
$$\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \overline{\alpha} \langle \mathbf{x}, \mathbf{z} \rangle + \overline{\beta} \langle \mathbf{y}, \mathbf{z} \rangle$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$.[1]

(b) $\|\mathbf{x}\|$ is a norm.

(c) The inner product and associated norm obeys

(i) (Cauchy–Schwarz inequality) $\big| \langle \mathbf{x}, \mathbf{y} \rangle \big| \leq \|\mathbf{x}\| \; \|\mathbf{y}\|$

(ii) (Parallelogram law) $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$

(iii) (Polarization identities)

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \tfrac{1}{2}\big\{ \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 \big\} + \tfrac{1}{2i}\big\{ \|\mathbf{x} + i\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 \big\} \\ &= \tfrac{1}{4}\big\{ \|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 \big\} + \tfrac{1}{4i}\big\{ \|\mathbf{x} + i\mathbf{y}\|^2 - \|\mathbf{x} - i\mathbf{y}\|^2 \big\} \end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$

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[1] Physicists and mathematical physicists generally use the convention that inner products are linear in the second argument and conjugate linear in the first. Some mathematicians use the convention that inner products are linear in the first argument and conjugate linear in the second.

Lemma 6 *Let $\| \cdot \|$ be a norm on a vector space $\mathcal{V}$. There exists an inner product $\langle \cdot , \cdot \rangle$ on $\mathcal{V}$ such that*

$$\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2 \qquad \textit{for all } \mathbf{x} \in \mathcal{V}$$

if and only if $\| \cdot \|$ obeys the parallelogram law.

Definition 7 (Banach Space)

(a) A Banach space is a complete normed vector space.

(b) Two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be isometric if there exists a map $U : \mathcal{B}_1 \to \mathcal{B}_2$ that is

(i) linear (meaning that $U(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha U(\mathbf{x})+\beta U(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{B}_1$ and $\alpha, \beta \in \mathbb{C}$)

(ii) onto (a.k.a. surjective)

(iii) isometric (meaning that $\|U\mathbf{x}\|_{\mathcal{B}_2} = \|\mathbf{x}\|_{\mathcal{B}_1}$ for all $\mathbf{x} \in \mathcal{B}_1$). This implies that U is 1–1 (a.k.a. injective).

Definition 8 (Hilbert Space)

(a) A Hilbert space $\mathcal{H}$ is a complex inner product space that is complete under the associated norm.

(b) Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be isomorphic (denoted $\mathcal{H}_1 \cong \mathcal{H}_2$) if there exists a map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that is

(i) linear

(ii) onto

(iii) inner product preserving (meaning that $\langle U\mathbf{x}, U\mathbf{y} \rangle_{\mathcal{H}_2} = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}_1}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}_1$)

Such a map is called *unitary*.

Lemma. *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces and if $U : \mathcal{H}_1 \to \mathcal{H}_2$ is linear, onto and isometric, then U is unitary.*

Theorem 9 (Completion) *If $\big(\mathcal{V}, \langle \cdot , \cdot \rangle_{\mathcal{V}}\big)$ is any inner product space, then there exists a Hilbert space $\big(\mathcal{H}, \langle \cdot, , \cdot \rangle_{\mathcal{H}}\big)$ and a map $U : \mathcal{V} \to \mathcal{H}$ such that*

(i) U is 1–1

(ii) U is linear

(iii) $\langle U\mathbf{x}, U\mathbf{y} \rangle_{\mathcal{H}} = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{V}}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$

(iv) $U(\mathcal{V}) = \big\{ U\mathbf{x} \;\big|\; \mathbf{x} \in \mathcal{V} \big\}$ is dense in $\mathcal{H}$. If $\mathcal{V}$ is complete, then $U(\mathcal{V}) = \mathcal{H}$.

$\mathcal{H}$ is called the completion of $\mathcal{V}$.

Example 10

(a) $\mathbb{C}^n = \left\{ \mathbf{x} = (x_1, \cdots x_n) \;\middle|\; x_1, \cdots x_n \in \mathbb{C} \right\}$ together with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{\ell=1}^{n} \bar{x}_\ell y_\ell$ is a Hilbert space.

(b) If $1 \le p < \infty$, then $\ell^p = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C},\ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$ together with the norm $\left\|(x_n)_{n\in\mathbb{N}}\right\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p}$ is a Banach space.

(c) $\ell^2 = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$ is a Hilbert space with the inner product $\left\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \right\rangle = \sum_{n=1}^{\infty} \overline{x_n} y_n$.

(d) $\ell^\infty = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \sup_n |x_n| < \infty \right\}$ and $c_0 = \left\{ (x_n)_{n\in\mathbb{N}} \;\middle|\; \lim_{n\to\infty} x_n = 0 \right\}$ are both Banach spaces with the norm $\left\|(x_n)_{n\in\mathbb{N}}\right\|_\infty = \sup_n |x_n|$.

Theorem Let $1 \le p, q, r \le \infty$ and let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$. Then
(a) (Minkowski inequality) If $\mathbf{x}, \mathbf{y} \in \ell^p$, then $\mathbf{x}+\mathbf{y} \in \ell^p$ and $\|\mathbf{x}+\mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$
(b) (Hölder inequality) If $\mathbf{x} \in \ell^p$ and $\mathbf{y} \in \ell^q$, then $(x_n y_n)_{n\in\mathbb{N}} \in \ell^r$ and we have
$\|(x_n y_n)_{n\in\mathbb{N}}\|_r \le \|\mathbf{x}\|_p \|\mathbf{y}\|_q$

(e) Let $\mathcal{X}$ be a metric space (or more generally a topological space) and

$$C(\mathcal{X}) = \left\{ f : \mathcal{X} \to \mathbb{C} \;\middle|\; f \text{ continuous, bounded} \right\}$$
$$C_0(\mathcal{X}) = \left\{ f : \mathcal{X} \to \mathbb{C} \;\middle|\; f \text{ continuous, compact support} \right\}$$

If $\mathcal{X}$ is a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ for some $n \in \mathbb{N}$, let

$$C_\infty(\mathcal{X}) = \left\{ f : \mathcal{X} \to \mathbb{C} \;\middle|\; f \text{ continuous}, \lim_{|x|\to\infty} f(x) = 0 \right\}$$

Then $C(\mathcal{X})$ and $C_\infty(\mathcal{X})$ are Banach spaces with the norm $\|f\| = \sup_{x\in\mathcal{X}} |f(x)|$. $C_0(\mathcal{X})$ is a normed vector space, but need not be complete.

(f) Let $1 \le p \le \infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space, with X a set, $\mathcal{M}$ a σ–algebra and μ a measure. For $p < \infty$, set

$$\mathcal{L}^p(X, \mathcal{M}, \mu) = \left\{ \varphi : X \to \mathbb{C} \;\middle|\; \varphi \text{ is } \mathcal{M}\text{–measurable and } \int |\varphi(x)|^p \, d\mu(x) < \infty \right\}$$
$$\|\varphi\|_p = \left[\int |\varphi(x)|^p \, d\mu(x) \right]^{1/p}$$

For $p = \infty$, set[(2)]

$$\mathcal{L}^\infty(X, \mathcal{M}, \mu) = \left\{ \varphi : X \to \mathbb{C} \;\middle|\; \varphi \text{ is } \mathcal{M}\text{–measurable and ess sup } |\varphi(x)| < \infty \right\}$$
$$\|\varphi\|_\infty = \text{ess sup } |\varphi(x)|$$

(2) The essential supremum of $|\varphi|$, with respect to the measure μ, is denoted ess $\sup_{x\in X} |\varphi(x)|$ and is defined by $\inf\{ a \ge 0 \mid |\varphi(x)| \le a$ almost everywhere, with respect to $\mu \}$.

This is not quite a Banach space because any function φ that is zero almost everywhere has "norm" zero. So we define an equivalence relation on $\mathcal{L}^p(X, \mathcal{M}, \mu)$ by

$$\varphi \sim \psi \iff \varphi = \psi \text{ a.e.}$$

As usual, the equivalence class of $\varphi \in \mathcal{L}^p(X, \mathcal{M}, \mu)$ is

$$[\varphi] = \left\{ \ \psi \in \mathcal{L}^p(X, \mathcal{M}, \mu) \ \middle| \ \psi \sim \varphi \ \right\}$$

Then

$$L^p(X, \mathcal{M}, \mu) = \left\{ \ [\varphi] \ \middle| \ \varphi \in \mathcal{L}^p(X, \mathcal{M}, \mu) \ \right\}$$

is a Banach space with

$$[\varphi] + [\psi] = [\varphi + \psi] \qquad a[\varphi] = [a\varphi] \qquad \big\|[\varphi]\big\|_p = \|\varphi\|_p$$

for all $\varphi, \psi \in \mathcal{L}^p(X, \mathcal{M}, \mu)$ and $a \in \mathbb{C}$, and $L^2(X, \mathcal{M}, \mu)$ is a Hilbert space with inner product

$$\langle [\varphi], [\psi] \rangle = \int \overline{\varphi(x)}\, \psi(x) \ d\mu(x)$$

for all $\varphi, \psi \in \mathcal{L}^2(X, \mathcal{M}, \mu)$. It is standard to write φ in place of $[\varphi]$.

If X is a Lebesgue measurable subset of $\mathbb{R}$, then $L^p(X)$ denotes the $L^p(X, \mathcal{M}, \mu)$ with $(\mathcal{M}, \mu)$ being Lebesgue measure.

Theorem Let $1 \le p, q, r \le \infty$.

(a) (Minkowski inequality) If $f, g \in L^p(X, \mathcal{M}, \mu)$, then $f + g \in L^p(X, \mathcal{M}, \mu)$ and $\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$

(b) (Hölder inequality) If $f \in L^p(X, \mathcal{M}, \mu)$ and $g \in L^q(X, \mathcal{M}, \mu)$, then we have $fg \in L^r(X, \mathcal{M}, \mu)$ and $\|fg\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}$

(g) Let D be an open subset of $\mathbb{C}$. Then

$$A^2(D) = \left\{ \ \varphi : D \to \mathbb{C} \ \middle| \ \varphi \text{ analytic}, \ \ \textstyle\int_D |\varphi(x + iy)|^2 \ dx\, dy < \infty \ \right\}$$

is a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle = \int_D \overline{\varphi(x + iy)}\, \psi(x + iy) \ dxdy$$

(h) Let $\ell \ge 0$ be an integer and Ω be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we shall use $\partial^\alpha \varphi(x)$ to denote the partial

derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \varphi(x)$. The order of this partial derivative is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define

$$\|\varphi\|_{\ell,\Omega} = \left\{ \sum_{|\alpha| \le \ell} \int_\Omega \left| \partial^\alpha \varphi(x) \right|^2 \, d^n x \right\}^{1/2}$$

for each $\varphi \in C^\ell(\Omega)$ for which the right hand side is finite. The Sobolev space $H^\ell(\Omega)$ is the completion of the vector space $\left\{ \varphi \in C^\ell(\Omega) \;\middle|\; \|\varphi\|_{\ell,\Omega} < \infty \right\}$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\ell,\Omega} = \sum_{|\alpha| \le \ell} \int_\Omega \overline{\partial^\alpha \varphi(x)} \, \partial^\alpha \psi(x) \, d^n x$$

Similarly, $H_0^\ell(\Omega)$ is the completion of $C_0^\infty(\Omega)$.

Theorem 11 *Let $-\infty < a < b < \infty$ and $1 \le p < \infty$. The following sets of functions are dense in $L^p\big([a,b]\big)$.*

(a) simple functions (functions of the form $\sum_{j=1}^n a_j \chi_{E_j}(x)$ with $n \in \mathbb{N}$ and the sets E_j measurable)

(b) step functions (functions of the form $\sum_{j=1}^n a_j \chi_{E_j}(x)$ with $n \in \mathbb{N}$ and the sets E_j intervals)

(c) continuous functions that vanish at a and b

(d) periodic C^∞ functions of period $b - a$

(e) C^∞ functions that are supported in (a, b)

Definition 12 (Basis) Let $\mathcal{B}$ be a Banach space and $\mathcal{H}$ a Hilbert space.

(a) A subset S of $\mathcal{H}$ is an *orthonormal subset* if each vector in S is of length one and each pair of distinct vectors in S is orthogonal.

(b) An *orthonormal basis* (or complete orthonormal system) for $\mathcal{H}$ is an orthonormal subset of $\mathcal{H}$, which is maximal in the sense that it is not properly contained in any other orthonormal subset of $\mathcal{H}$.

(c) A *Schauder basis* for $\mathcal{B}$ is a sequence $\big\{\mathbf{e}_n\big\}_{n\in\mathbb{N}}$ of elements of $\mathcal{B}$ such that for each $v \in \mathcal{B}$ there is a unique sequence $\big\{\alpha_n\big\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $v = \sum_{n=1}^\infty \alpha_n \mathbf{e}_n$.

(d) An *algebraic basis* (or Hamel basis) for $\mathcal{B}$ is a subset $\mathcal{S} \subset \mathcal{B}$ such that each $\mathbf{x} \in \mathcal{B}$ has a unique representation as a finite linear combination of elements of $\mathcal{S}$. This is the case if and only if every finite subset of $\mathcal{S}$ is linearly independent and each $\mathbf{x} \in \mathcal{B}$ has some representation as a finite linear combination of elements of $\mathcal{S}$.

Theorem 13 *Every Hilbert space has an orthonormal basis.*

Theorem 14 *Every vector space has an algebraic basis.*

Theorem 15 *Let $\{\mathbf{e}_i\}_{i\in\mathcal{I}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Then, for each $\mathbf{x}\in\mathcal{H}$, $\{\ i\in\mathcal{I}\ \big|\ \langle\mathbf{e}_i,\mathbf{x}\rangle\neq 0\ \}$ is countable*[3] *and*

$$\mathbf{x}=\sum_{i\in\mathcal{I}}\langle\mathbf{e}_i,\mathbf{x}\rangle\ \mathbf{e}_i \qquad \|\mathbf{x}\|^2=\sum_{i\in\mathcal{I}}|\,\langle\mathbf{e}_i,\mathbf{x}\rangle\,|^2$$

(The right hand sides converge independent of order.) Conversely, if $\{c_i\}_{i\in\mathcal{I}}\subset\mathbb{C}$ and $\sum_{i\in\mathcal{I}}|c_i|^2<\infty$, then $\sum_{i\in\mathcal{I}}c_i\,\mathbf{e}_i$ converges to an element of $\mathcal{H}$.

Example 16 For each $n\in\mathbb{Z}$, set

$$\mathbf{e}_n(x)=\tfrac{1}{\sqrt{2\pi}}e^{inx}$$

Then $\{\mathbf{e}_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2\big([0,2\pi]\big)$.

Definition 17 (Separable) A metric space is said to be separable if it has a countable dense subset.

Lemma 18 *A metric space $\big(\mathcal{M},d\big)$ fails to be separable if and only if there is an $\varepsilon>0$ and an uncountable subset $\{m_i\}_{i\in\mathcal{I}}\subset$ with $d(m_i,m_j)\geq\varepsilon$ for all $i,j\in\mathcal{I}$ with $i\neq j$.*

Theorem 19 *Let $\mathcal{H}$ be a Hilbert space.*

(a) $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis.

(b) If $\dim\mathcal{H}=n\in\mathbb{N}$, then $\mathcal{H}\cong\mathbb{C}^n$.

(c) If $\mathcal{H}$ is separable but is not of finite dimension, then $\mathcal{H}\cong\ell^2$.

Example 20

(a) As $L^2([0,2\pi])$ has a countable, orthonormal basis, it is separable and isomorphic to ℓ^2.

(b) ℓ^∞ is not separable. To see this define, for each subset $S\subset\mathbb{N}$, $x^{(S)}=\big(x_n^{(S)}\big)_{n\in\mathbb{N}}\in\ell^\infty$ by

$$x_n^{(S)}=\begin{cases}1 & \text{if } n\in S\\ 0 & \text{if } n\notin S\end{cases}$$

This is an uncountable family of elements of ℓ^∞ with $\|x^{(S)}-x^{(T)}\|_\infty=1$ for all distinct subsets S,T of $\mathbb{N}$.

[3] We'll include finite in countable.

Definition 21 (Orthogonal Complement) The orthogonal complement, $\mathcal{M}^\perp$, of any subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, is defined to be

$$\mathcal{M}^\perp = \left\{ \mathbf{y} \in \mathcal{H} \;\middle|\; \langle \mathbf{y}, \mathbf{x} \rangle = 0 \text{ for all } \mathbf{x} \in \mathcal{M} \right\}$$

Theorem 22 *Let $\mathcal{M}$ be a linear subspace of a Hilbert space $\mathcal{H}$. Then*

(a) $\mathcal{M}^\perp$ is a closed linear subspace of $\mathcal{H}$.

(b) $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$

(c) $\left(\mathcal{M}^\perp\right)^\perp = \overline{\mathcal{M}}$ (the closure of $\mathcal{M}$)

Theorem 23 (Projection) *Let $\mathcal{M}$ be a closed linear subspace of a Hilbert space $\mathcal{H}$. Then each $\mathbf{x} \in \mathcal{H}$ has a unique representation $\mathbf{x} = \mathbf{x}^\parallel + \mathbf{x}^\perp$ with $\mathbf{x}^\parallel \in \mathcal{M}$ and $\mathbf{x}^\perp \in \mathcal{M}^\perp$.*

Definition 24 (Linear Operator) Let $\mathcal{B}$, $\mathcal{B}'$ be Banach spaces and $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces.

(a) Let $\mathcal{D}$ be a linear subspace of $\mathcal{B}$. A map $A : \mathcal{D} \to \mathcal{B}'$ is called a *linear operator* if it obeys

$$A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A(\mathbf{x}) + \beta A(\mathbf{y}) \qquad \text{for all } \alpha, \beta \in \mathbb{C} \text{ and } \mathbf{x}, \mathbf{y} \in \mathcal{D}$$

One usually denotes the image of $\mathbf{x}$ under A as $A\mathbf{x}$, rather than $A(\mathbf{x})$. The set $\mathcal{D}$ is called the domain of A and is generally denoted $\mathcal{D}(A)$. One often calls A a "linear operator on $\mathcal{B}$" even when its domain is a proper subset of $\mathcal{B}$.

(b) A linear operator $A : \mathcal{D} \to \mathcal{B}'$ is said to be *bounded* if

$$\|A\| = \sup_{\mathbf{0} \neq \mathbf{x} \in \mathcal{D}} \frac{\|A\mathbf{x}\|_{\mathcal{B}'}}{\|\mathbf{x}\|_{\mathcal{B}}} = \sup_{\substack{\mathbf{x} \in \mathcal{D} \\ \|\mathbf{x}\|_{\mathcal{B}} = 1}} \|A\mathbf{x}\|_{\mathcal{B}'} \tag{1}$$

is finite. The set of all bounded, linear operators having domain $\mathcal{B}$ and taking values in $\mathcal{B}'$ is denoted $\mathcal{L}(\mathcal{B}, \mathcal{B}')$. With the norm (1), it is itself a Banach space. The set of all bounded, linear operators defined on $\mathcal{B}$ and taking values in $\mathcal{B}$ is denoted $\mathcal{L}(\mathcal{B})$.

(c) A *linear functional* on $\mathcal{B}$ is a linear operator $f : \mathcal{B} \to \mathbb{C}$. A *bounded linear functional* on $\mathcal{B}$ is a linear operator $f : \mathcal{B} \to \mathbb{C}$ for which

$$\sup_{\mathbf{0} \neq \mathbf{x} \in \mathcal{B}} \frac{|f(\mathbf{x})|}{\|\mathbf{x}\|_{\mathcal{B}}}$$

is finite.

(d) The *dual space* of a Banach space $\mathcal{B}$ is the space $\mathcal{B}^*$ of all bounded linear functionals on $\mathcal{B}$. The dual space is itself a Banach space.

(e) Let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ be a linear operator. Denote

$$\mathcal{D}(T^*) = \left\{ \ \varphi \in \mathcal{H}' \ \middle| \ \exists! \ \eta \in \mathcal{H} \ \text{ s.t. } \ \langle T\psi, \varphi \rangle_{\mathcal{H}'} = \langle \psi, \eta \rangle_{\mathcal{H}} \quad \forall \ \ \psi \in \mathcal{D}(T) \ \right\}$$

If $\varphi \in \mathcal{D}(T^*)$ the corresponding η is denoted $T^*\varphi$. Thus $T^*\varphi$ is the unique vector in $\mathcal{H}$ such that

$$\langle T\psi, \varphi \rangle_{\mathcal{H}'} = \langle \psi, T^*\varphi \rangle_{\mathcal{H}} \qquad \text{for all } \psi \in \mathcal{D}(T)$$

The operator T^* is called the adjoint of T.

Proposition 25 *The normed vector space $\mathcal{L}(\mathcal{B}, \mathcal{B}')$, with the norm (1), is a Banach space.*

Lemma 26 *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then there is a linear operator $W : \mathcal{H} \to \mathcal{H}$ which is defined on all of $\mathcal{H}$, but is* **not** *bounded.*

Example 27

(a) *Matrices*: Let $n \in \mathbb{N}$. An $n \times n$ matrix $\left[M_{i,j}\right]_{1 \le i,j \le n}$ is naturally associated to the operator $M : \mathbb{C}^n \to \mathbb{C}^n$ determined by

$$(M\mathbf{x})_i = \sum_{j=1}^{n} M_{i,j} x_j$$

The adjoint operator is the operator so associated to the matrix $\left[M^*_{i,j} = \overline{M_{j,i}}\right]_{1 \le i,j \le n}$.

(b) *Multiplication Operators*: Let $1 \le p < \infty$, let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$ be measurable. If the essential supremum of f is finite, then

$$\begin{aligned} M_f : L^p(X, \mathcal{M}, \mu) &\to L^p(X, \mathcal{M}, \mu) \\ \varphi(x) &\mapsto (f\varphi)(x) = f(x)\,\varphi(x) \end{aligned}$$

is a bounded linear operator with $\|M_f\| = \operatorname*{ess\,sup}_{x \in X} |f(x)|$. On the other hand, if the essential supremum of f is infinite, then M_f will not be defined on all of $L^p(X, \mathcal{M}, \mu)$ and will not be bounded (as a map into $L^p(X, \mathcal{M}, \mu)$). In the case $p = 2$, $M_f^* = M_{\overline{f}}$.

(c) *Projection Operators*: Let $\mathcal{H}$ be Hilbert space and let $\mathcal{M}$ be a nonempty, closed, linear subspace of $\mathcal{H}$. Define the map $P : \mathcal{H} \to \mathcal{H}$ by

$$P\mathbf{x} = \mathbf{x}^{\|} \qquad \text{where } \mathbf{x} = \mathbf{x}^{\perp} + \mathbf{x}^{\|} \text{ is the decomposition of Theorem 23.a}$$

It is a bounded linear operator with $\|P\| = 1$, called the orthogonal projection on $\mathcal{M}$. It obeys

$$P^2 = P \qquad P^* = P$$

where P^* is the adjoint of P. Conversely if $P : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator that obeys $P^2 = P$ and $P^* = P$, then P is orthogonal projection on $\mathcal{M} = \text{range}(P)$.

(d) *Integral Operators*: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces and $T : X \times Y \to \mathbb{C}$ be a function that is measurable with respect to $\mathcal{M} \otimes \mathcal{N}$. Let $1 \le p \le \infty$ and $\varphi \in L^p(Y, \mathcal{N}, \nu)$. Define, for each $x \in X$ for which the function $y \mapsto T(x, y)\varphi(y)$ is in $L^1(Y, \mathcal{N}, \nu)$,

$$(T\varphi)(x) = \int_Y T(x, y)\varphi(y)\, d\nu(y) \tag{2}$$

(i) If

$$M_1 = \operatorname*{ess\,sup}_{x \in X} \int_Y |T(x, y)|\, d\nu(y) < \infty$$
$$M_2 = \operatorname*{ess\,sup}_{y \in Y} \int_X |T(x, y)|\, d\mu(x) < \infty$$

then (2) defines a bounded operator $T : L^p(Y, \mathcal{N}, \nu) \to L^p(X, \mathcal{M}, \mu)$ with norm $\|T\| \le M_1^{1-\frac{1}{p}} M_2^{\frac{1}{p}}$.

(ii) If the Hilbert–Schmidt norm

$$\|T\|_{\text{H.S.}} = \Big[\int_{X \times Y} |T(x, y)|^2\, d\mu \times \nu(x, y) \Big]^{1/2}$$

is finite, then (2) defines a bounded operator $T : L^2(Y, \mathcal{N}, \nu) \to L^2(X, \mathcal{M}, \mu)$ with norm $\|T\| \le \|T\|_{\text{H.S.}}$.

In the case $p = 2$,

$$(T^*\psi)(y) = \int_x \overline{T(x, y)}\psi(x)\, d\mu(x)$$

(e) *Differential Operators*: Let Ω be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Recall that if $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we use $\partial^\alpha u(x)$ to denote the partial derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ to denote the order of this partial derivative. For any finite subset $\mathcal{I} \subset \mathbb{N}_0^n$ and any family $\big\{a_\alpha(x)\big\}_{\alpha \in \mathcal{I}}$ of bounded, measurable functions on Ω the map

$$\varphi(x) \mapsto \sum_{\alpha \in \mathcal{I}} a_\alpha(x)\, \partial^\alpha \varphi(x)$$

is a linear map on $C^\infty(\Omega) \cap L^2(\Omega)$. But it is not bounded as a map from $L^2(\Omega)$ to $L^2(\Omega)$.

Lemma 28 *Let $\mathcal{H}$ be a Hilbert space. Let $P : \mathcal{H} \to \mathcal{H}$ be a bounded operator that obeys*

$$P^2 = P \qquad P^* = P$$

Then P is orthogonal projection on the range of P.

Lemma 29 *Let P and P' be orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Then $P + P'$ is again an orthogonal projection if and only if $PP' = P'P = 0$.*

Theorem 30 *Let $\mathcal{V}$ and $\mathcal{V}'$ be two normed vector spaces and let $T : \mathcal{V} \to \mathcal{V}'$ be a linear transformation. The following are equivalent.*
(i) T is continuous at every $\mathbf{x} \in \mathcal{V}$.
(ii) T is continuous at one $\mathbf{x}_0 \in \mathcal{V}$.
(iii) T is bounded.

Theorem 31 (Hahn–Banach corollaries) *Let $\mathcal{B}$ be a Banach space.*

(a) Let $\mathcal{S}$ be a subspace of $\mathcal{B}$ and $\lambda \in \mathcal{S}^*$. Then there is a $\Lambda \in \mathcal{B}^*$ such that $\|\Lambda\|_{\mathcal{B}^*} = \|\lambda\|_{\mathcal{S}^*}$ and $\Lambda(\mathbf{x}) = \lambda(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{S}$.

(b) Let $\mathbf{x} \in \mathcal{B}$ and $\zeta \in \mathbb{C}$. There is a $\Lambda \in \mathcal{B}^*$ such that $\Lambda(\mathbf{x}) = \zeta\|\mathbf{x}\|_{\mathcal{B}}$ and $\|\Lambda\|_{\mathcal{B}^*} = |\zeta|$.

(c) Let $\mathcal{Y}$ be a subspace of $\mathcal{B}$ and $\mathbf{x} \in \mathcal{B}$ with the distance from $\mathbf{x}$ to $\mathcal{Y}$ being d. There is a $\Lambda \in \mathcal{B}^*$ such that $\|\Lambda\|_{\mathcal{B}^*} \leq 1$, $\Lambda(\mathbf{x}) = d$ and $\Lambda(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{Y}$.

(d) Let $\mathbf{x} \in \mathcal{B}$. Then

$$\|\mathbf{x}\|_{\mathcal{B}} = \sup_{\substack{\Lambda \in \mathcal{B}^* \\ \|\Lambda\|_{\mathcal{B}^*} = 1}} \big|\Lambda(\mathbf{x})\big|$$

Theorem 32 (The B.L.T. Theorem) *Let $\mathcal{V}$ be a dense linear subspace of a Banach space $\mathcal{B}$. Let $\mathcal{B}'$ be a second Banach space and $T : \mathcal{V} \to \mathcal{B}'$ be a bounded linear transformation. Then there is a unique bounded linear transformation $\tilde{T} : \mathcal{B} \to \mathcal{B}'$ such that $T\mathbf{x} = \tilde{T}\mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$. Furthermore $\|T\| = \|\tilde{T}\|$.*

Example 33 We define the Fourier transform as a unitary operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. To start we define Schwartz space to be

$$\mathcal{S}(\mathbb{R}) = \left\{ \ \varphi : \mathbb{R} \to \mathbb{C} \ \middle| \ \varphi \text{ is } C^\infty, \ \|\varphi\|_{n,m} < \infty \text{ for all integers } n, m \geq 0 \ \right\}$$

where $\|\varphi\|_{n,m} = \sup\limits_{x \in \mathbb{R}} \left| x^n \frac{d^m \varphi}{dx^m}(x) \right|$

Next we define the Fourier transform and inverse Fourier transform on $\mathcal{S}(\mathbb{R})$ by

$$\hat{\varphi}(k) = \int_{-\infty}^{\infty} e^{-ikx}\varphi(x)\ dx$$
$$\check{\psi}(x) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\psi(k)\ dk$$

and verify that the linear functions $\varphi \mapsto \hat{\varphi}$ and $\psi \mapsto \check{\psi}$ each map $\mathcal{S}(\mathbb{R})$ into (in fact onto) $\mathcal{S}(\mathbb{R})$ and are inverses of each other and obey

$$\int_{-\infty}^{\infty} \overline{\varphi(x)}\,\psi(x)\ dx = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{\varphi}(k)}\,\hat{\psi}(k)\ dk$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$. Then the B.L.T. theorem provides us with the unique bounded extension of the map $\varphi \mapsto \hat{\varphi}$ to $L^2(\mathbb{R})$, which we call $\mathcal{F}$. For the details, see the notes "Tempered Distributions and the Fourier Trransform".

Theorem 34 (Riesz Representation Theorem) *Let $\mathcal{H}$ be a Hilbert space and $\lambda \in \mathcal{H}^*$ be a bounded linear functional on $\mathcal{H}$. Then there is a unique $\mathbf{y}_\lambda \in \mathcal{H}$ such that*

$$\lambda(\mathbf{x}) = \langle \mathbf{y}_\lambda, \mathbf{x}\rangle$$

for all $\mathbf{x} \in \mathcal{H}$. Furthermore $\|\lambda\|_{\mathcal{H}^*} = \|\mathbf{y}_\lambda\|_{\mathcal{H}}$.

Corollary 35 *Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ and $C \geq 0$ obey*

(i) $B(\mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z}) = \alpha B(\mathbf{x}, \mathbf{y}) + \beta B(\mathbf{x}, \mathbf{z})$

(ii) $B(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \bar{\alpha} B(\mathbf{x}, \mathbf{z}) + \bar{\beta} B(\mathbf{y}, \mathbf{z})$

(iii) $\big|B(\mathbf{x}, \mathbf{y})\big| \leq C\|\mathbf{x}\|\,\|\mathbf{y}\|$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. Then there is a unique $A \in \mathcal{L}(\mathcal{H})$ such that $B(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, A\mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Furthermore $\|A\| \leq C$.

Corollary 36 *Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces and $T : \mathcal{H} \to \mathcal{H}'$ be a bounded linear operator. Then the adjoint T^* of T is a bounded linear operator defined on all of $\mathcal{H}'$.*

Definition 37 (Operator Topologies) Let $\mathcal{B}$ and $\mathcal{B}'$ be Banach spaces. Let $T : \mathcal{B} \to \mathcal{B}'$ and, for each $n \in \mathbb{N}$, $T_n : \mathcal{B} \to \mathcal{B}'$ be bounded linear operators.

(a) The sequence of $\{T_n\}_{n\in\mathbb{N}}$ of operators is said to converge *uniformly* or *in norm* to T if

$$\lim_{n\to\infty} \|T - T_n\| = 0$$

(b) The sequence of $\{T_n\}_{n\in\mathbb{N}}$ of operators is said to converge *strongly* to T if

$$\lim_{n\to\infty} \|T\mathbf{x} - T_n\mathbf{x}\|_{\mathcal{B}'} = 0 \qquad \text{for each } \mathbf{x} \in \mathcal{B}$$

(c) The sequence of $\{T_n\}_{n\in\mathbb{N}}$ of operators is said to converge *weakly* to T if

$$\lim_{n\to\infty} \ell\big(T_n\mathbf{x}\big) = \ell\big(T\mathbf{x}\big) \qquad \text{for each } \mathbf{x} \in \mathcal{B} \text{ and each } \ell \in \mathcal{B}'^*$$

In the event that $\mathcal{B}'$ is a Hilbert space, this is equivalent to

$$\lim_{n\to\infty} \langle \mathbf{y}, T_n\mathbf{x}\rangle_{\mathcal{B}'} = \langle \mathbf{y}, T\mathbf{x}\rangle_{\mathcal{B}'} \qquad \text{for each } \mathbf{x} \in \mathcal{B} \text{ and each } \mathbf{y} \in \mathcal{B}'$$

Remark 38 (Operator Topologies) Since $\big|\ell\big((T_n - T)\mathbf{x}\big)\big| \le \|\ell\|_{\mathcal{B}'^*}\big\|(T_n - T)\mathbf{x}\big\|_{\mathcal{B}'}$ and $\big\|(T_n - T)\mathbf{x}\big\|_{\mathcal{B}'} \le \|T_n - T\|\,\|\mathbf{x}\|_{\mathcal{B}}$,

$$\text{norm convergence} \implies \text{strong convergence} \implies \text{weak convergence}$$

In general the other implications are false, unless $\mathcal{B}$ and $\mathcal{B}'$ are finite dimensional. This is illustrated by the following

Example 39 (Operator Topologies) Let $\mathcal{B} = \mathcal{B}' = \ell^2$.

(a) Let

$$P_n\big(x_1, x_2, x_3 \cdots\big) = \big(\overbrace{0, \cdots, 0}^{n \text{ places}}, x_{n+1}, x_{n+2}, x_{n+3}, \cdots\big)$$

be projection on the orthogonal complement of the first n components. Then for each fixed $\mathbf{x} \in \ell^2$, $\lim_{n\to\infty} P_n\mathbf{x} = \mathbf{0}$ so that P_n converges strongly to 0 as $n \to \infty$. But, for any $n > m$,

$$(P_m - P_n)\big(x_1, x_2, x_3 \cdots\big) = \big(\overbrace{0, \cdots, 0}^{m \text{ places}}, x_{m+1}, x_{m+2}, \cdots, x_n, 0, \cdots\big)$$

so that there is a vector $\mathbf{x} \in \ell^2$ with $(P_m - P_n)\mathbf{x} = \mathbf{x}$. Consequently $\|P_m - P_n\| = 1$, the sequence is not Cauchy and does not converge in norm.

(b) Let

$$R_n\big(x_1, x_2, x_3 \cdots\big) = \big(\overbrace{0, \cdots, 0}^{n \text{ places}}, x_1, x_2, x_3, \cdots\big)$$

be right shift by n places. For any $\mathbf{x}, \mathbf{y} \in \ell^2$

$$\big|\,\langle \mathbf{y}, R_n\mathbf{x}\rangle\,\big| = \big|\,\langle P_n\mathbf{y}, R_n\mathbf{x}\rangle\,\big| \le \|P_n\mathbf{y}\|\,\|R_n\mathbf{x}\| = \|P_n\mathbf{y}\|\,\|\mathbf{x}\| \overset{n\to\infty}{\longrightarrow} 0$$

So R_n converges weakly to zero as $n \to \infty$. On the other hand, $\|R_n\mathbf{x}\| = \|\mathbf{x}\|$ for all $n \in \mathbb{N}$ and $\mathbf{x} \in \ell^2$. So the R_n does not converge strongly or in norm. (If R_n did converge either strongly or in norm to some R, the fact that $R_n \overset{\text{weakly}}{\longrightarrow} 0$ would force $R = 0$.)

Theorem 40 (Adjoints) *Let $\mathcal{H}$ be a Hilbert space and $S, T \in \mathcal{L}(\mathcal{H})$.*

(a) The map $A \mapsto A^*$ is a conjugate linear isometric isomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})$. In particular

$$(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^* \qquad \|A^*\| = \|A\|$$

for all $A, B \in \mathcal{L}(\mathcal{H})$ and all $\alpha, \beta \in \mathbb{C}$.

(b) $(TS)^* = S^* T^*$

(c) $(T^*)^* = T$

(d) If T has a bounded inverse, then T^* has a bounded inverse and $(T^*)^{-1} = (T^{-1})^*$.

(e) The map $A \mapsto A^*$ is continuous in the weak and uniform topologies. That is, if $\{A_n\}_{n\in\mathbb{N}}$ converges to A weakly (in norm), then $\{A_n^*\}_{n\in\mathbb{N}}$ converges to A^* weakly (in norm). The map $A \mapsto A^*$is continuous in the strong topology if and only if $\mathcal{H}$ is finite dimensional.

(f) $\|T^*T\| = \|T\|^2$

(g) If $T = T^*$, then $\|T\| = \sup\left\{ \ |\langle T\mathbf{x}, \mathbf{x}\rangle| \ \middle| \ \mathbf{x} \in \mathcal{H}, \ \|\mathbf{x}\| = 1 \ \right\}$.

Example 41 Let $\mathcal{H} = \ell^2$ and define the right and left shift operators by

$$L(x_1, x_2, x_3, \cdots) = (x_2, x_3, \cdots)$$
$$R(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$$

First observe that $\|L\| = \|R\| = 1$ and that

$$\langle \mathbf{y}, L\mathbf{x}\rangle = \sum_{j=1}^{\infty} \overline{y_j}\,(L\mathbf{x})_j = \sum_{j=1}^{\infty} \overline{y_j}\, x_{j+1} = \sum_{i=2}^{\infty} \overline{y_{i-1}}\, x_i = \sum_{i=1}^{\infty} \overline{(R\mathbf{y})_i}\, x_i = \langle R\mathbf{y}, \mathbf{x}\rangle$$

so that $L^* = R$ and $R^* = L$. Next observe that, for each $n \in \mathbb{N}$ and $\mathbf{x} \in \ell^2$,

$$\|L^n\mathbf{x}\|^2 = \sum_{m=n+1}^{\infty} |x_m|^2 \overset{n\to\infty}{\longrightarrow} 0$$
$$\|R^n\mathbf{x}\|^2 = \sum_{m=1}^{\infty} |x_m|^2 \quad = \|\mathbf{x}\|^2$$

Thus, as $n \to \infty$, L^n converges strongly to zero, but $L^{n*} = R^n$ does not converge strongly to anything. On the other hand, L^{n*} does converge weakly to zero since, for all $\mathbf{x}, \mathbf{y} \in \ell^2$,

$$\left|\langle \mathbf{y}, L^{n*}\mathbf{x}\rangle\right| = \left|\langle \mathbf{y}, R^n\mathbf{x}\rangle\right| = \left|\langle L^n\mathbf{y}, \mathbf{x}\rangle\right| \le \|L^n\mathbf{y}\|\,\|\mathbf{x}\| \overset{n\to\infty}{\longrightarrow} 0$$

Proposition 42 *Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator.*

(a) We have

$$\|T\| = \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathcal{H}\\ \mathbf{x},\mathbf{y}\neq\mathbf{0}}} \frac{|\langle \mathbf{y}, T\mathbf{x}\rangle|}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$$

(b) Assume in addition that $T = T^*$. Then

$$\|T\| = \sup_{\substack{\mathbf{x}\in\mathcal{H}\\ \mathbf{x}\neq\mathbf{0}}} \frac{|\langle \mathbf{x}, T\mathbf{x}\rangle|}{\|\mathbf{x}\|^2}$$

Example 43 For part (b) of Proposition 42 it is critical that $T = T^*$, even in finite dimensions. For example, the 2×2 matrix

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has norm 1. In particular it maps the unit vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to the unit vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. But, for any unit vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$\left| [\,\overline{x_1} \quad \overline{x_2}\,] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right| = \left|\overline{x_1} x_2\right| \le \frac{1}{2}\big(|x_1|^2 + |x_2|^2\big) \le \frac{1}{2}$$

Theorem 44 (Principle of Uniform Boundedness etc.) *Unless otherwise stated, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is linear and has domain $\mathcal{X}$.*

(a) T is bounded if and only if

$$T^{-1}\{\ \mathbf{y} \in \mathcal{Y} \mid \|\mathbf{y}\|_{\mathcal{Y}} \le 1\ \} = \{\ \mathbf{x} \in \mathcal{X} \mid \|T\mathbf{x}\|_{\mathcal{Y}} \le 1\ \}$$

has nonempty interior. ($\mathcal{X}, \mathcal{Y}$ need not be complete.)

(b) Principle of Uniform Boundedness: *Let $\mathcal{F} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$.*

If, for each $\mathbf{x} \in \mathcal{X}$, $\{\ \|T\mathbf{x}\| \mid T \in \mathcal{F}\ \}$ *is bounded,*
then $\{\ \|T\| \mid T \in \mathcal{F}\ \}$ *is bounded,*

($\mathcal{Y}$ need not be complete.)

(c) If $B : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ is bilinear and continuous in each variable separately (i.e. $B(\mathbf{x}, \mathbf{y})$ is continuous in $\mathbf{x}$ for each fixed $\mathbf{y}$ and vice versa), then $B(\mathbf{x}, \mathbf{y})$ is jointly continuous (i.e. if $\lim_{n\to\infty} \mathbf{x}_n = 0$ and $\lim_{n\to\infty} \mathbf{y}_n = 0$, then $\lim_{n\to\infty} B(\mathbf{x}_n, \mathbf{y}_n) = 0$.

(d) Open Mapping Theorem: *If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is surjective (i.e. onto) and if $\mathcal{O}$ is an open subset of $\mathcal{X}$, then $T\mathcal{O} = \left\{ T\mathbf{x} \;\middle|\; \mathbf{x} \in \mathcal{O} \right\}$ is an open subset of $\mathcal{Y}$.*

(e) Inverse Mapping Theorem: *If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is bijective (i.e. 1–1 and onto), then T^{-1} is bounded.*

(f) Closed Graph Theorem: *The graph of T is defined to be*

$$\Gamma(T) = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y} \;\middle|\; \mathbf{y} = T\mathbf{x} \right\}$$

Then

$$T \text{ is bounded} \iff \Gamma(T) \text{ is closed}$$

In other words, T is bounded if and only if

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}, \;\; \lim_{n\to\infty} T\mathbf{x}_n = \mathbf{y} \implies \mathbf{y} = T\mathbf{x}$$

(g) Hellinger–Toeplitz Theorem: *Let T be an everywhere defined linear operator on the Hilbert space $\mathcal{H}$ that obeys $\langle \mathbf{x}, T\mathbf{y} \rangle = \langle T\mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Then T is bounded.*

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