

Compact Operators

In these notes we provide an introduction to compact linear operators on Banach and Hilbert spaces. These operators behave very much like familiar finite dimensional matrices, without necessarily having finite rank. For more thorough treatments, see [RS, Y].

Definition 1 Let \mathcal{X} and \mathcal{Y} be Banach spaces. A linear operator $C : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be compact if for each bounded sequence $\{x_i\}_{i \in \mathbb{N}} \subset \mathcal{X}$, there is a subsequence of $\{Cx_i\}_{i \in \mathbb{N}}$ that is convergent.

Example 2 Let $a < b$ and $c < d$. If $C : [c, d] \times [a, b] \rightarrow \mathbb{C}$ is continuous, then the integral operator

$$(Cf)(y) = \int_a^b C(y, x)f(x) dx$$

is compact as an operator from $\mathcal{X} = C[a, b]$, the space of continuous functions on $[a, b]$ with supremum norm, to $\mathcal{Y} = C[c, d]$.

Problem 1 Use the Arzelà–Ascoli theorem to prove that the operator C of Example 2 is compact.

Example 3 (Hilbert–Schmidt Operators) Let $\langle X, \mu \rangle$ and $\langle Y, \nu \rangle$ be measure spaces and let $k(x, y)$ be a measurable function on $X \times Y$ with

$$\int_{X \times Y} |k(x, y)|^2 d\mu(x)d\nu(y) < \infty$$

Then

$$(Kf)(x) = \int_Y k(x, y)f(y) d\nu(y)$$

is a compact map from $L^2(Y, d\nu)$ to $L^2(X, d\mu)$. Such an operator is called Hilbert–Schmidt.

Proof: Let $\{f_i\}_{i \in \mathbb{N}}$ be a bounded sequence in $L^2(Y, d\nu)$. By part (c) of Problem 3, below, $\{f_i\}_{i \in \mathbb{N}}$ has a weakly convergent subsequence. By throwing away all but this subsequence, we may assume that $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to $f \in L^2(Y, d\nu)$.

We now show that $\{Kf_i\}_{i \in \mathbb{N}}$ converges strongly to $Kf \in L^2(X, d\mu)$. Since $\int_{X \times Y} |k(x, y)|^2 d\mu(x)d\nu(y) < \infty$ we have that $\int_Y |k(x, y)|^2 d\nu(y) < \infty$ for almost every

$x \in X$. For any such $x \in X$,

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_Y k(x, y) f_i(y) \, d\nu(y) &= \lim_{i \rightarrow \infty} \langle \overline{k(x, \cdot)}, f_i \rangle_{L^2(Y, d\nu)} = \langle \overline{k(x, \cdot)}, f \rangle_{L^2(Y, d\nu)} \\ &= \int_Y k(x, y) f(y) \, d\nu(y) \end{aligned}$$

Furthermore, by Cauchy–Schwarz,

$$\begin{aligned} |(Kf_i)(x)| &\leq \int_Y |k(x, y) f_i(y)| \, d\nu(y) \leq \|f_i\|_{L^2(Y, d\nu)} \sqrt{\int_Y |k(x, y)|^2 \, d\nu(y)} \\ &\leq \sup_i \|f_i\|_{L^2(Y, d\nu)} \sqrt{\int_Y |k(x, y)|^2 \, d\nu(y)} \equiv H(x) \end{aligned}$$

Thus we have shown that $(Kf_i)(x)$ converges pointwise to $(Kf)(x)$ for almost every x and is bounded, for all i by the function $H(x)$ which is square integrable with respect to $d\mu(x)$. Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{i \rightarrow \infty} \|Kf - Kf_i\|_{L^2(X, d\mu)}^2 = \lim_{i \rightarrow \infty} \int_X |(Kf)(x) - (Kf_i)(x)|^2 \, d\mu(x) = 0$$

■

Problem 2 Prove that any Hilbert–Schmidt operator is bounded.

Problem 3 Let \mathcal{H} be a Hilbert Space. A sequence $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{H}$ is said to converge weakly to $f \in \mathcal{H}$ if

$$\lim_{i \rightarrow \infty} \langle g, f_i \rangle = \langle g, f \rangle$$

for all $g \in \mathcal{H}$.

(a) Give an example of a sequence that converges weakly but not strongly.

(b) Prove that if $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to f , then $\|f\| \leq \liminf_{i \rightarrow \infty} \|f_i\|$. Prove that if $\{f_i\}_{i \in \mathbb{N}}$ converges weakly to f and $\|f\| = \lim_{i \rightarrow \infty} \|f_i\|$, then $\{f_i\}_{i \in \mathbb{N}}$ converges strongly to f .

(c) Prove that \mathcal{H} is weakly sequentially compact. That is, every bounded sequence in \mathcal{H} has a weakly convergent subsequence.

Example 4 (Nuclear Operators) Let \mathcal{X} and \mathcal{Y} be Banach spaces and denote by \mathcal{X}' the dual space of \mathcal{X} . That is, the space of bounded linear functionals on \mathcal{X} . If $\{x'_i\}_{i \in \mathbb{N}}$ is a bounded sequence in \mathcal{X}' , $\{y_i\}_{i \in \mathbb{N}}$ is a bounded sequence in \mathcal{Y} and $\{c_i\}_{i \in \mathbb{N}}$ is a set of complex numbers obeying $\sum_i |c_i| < \infty$, then

$$Kx = \sum_{i=1}^{\infty} c_i x'_i(x) y_i$$

is called a nuclear operator from \mathcal{X} to \mathcal{Y} . Since

$$\sum_{i=1}^{\infty} |c_i| |x'_i(x)| \|y_i\|_{\mathcal{Y}} \leq \|x\|_{\mathcal{X}} \sup_i \|y_i\|_{\mathcal{Y}} \sup_i \|x'_i\|_{\mathcal{X}'} \sum_{i=1}^{\infty} |c_i|$$

the series defining Kx converges strongly and K is a bounded operator of norm at most $\sup_i \|y_i\|_{\mathcal{Y}} \sup_i \|x'_i\|_{\mathcal{X}'} \sum_{i=1}^{\infty} |c_i|$.

Problem 4 Prove that any nuclear operator is compact.

Proposition 5 Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be Banach spaces.

- (a) If $C : \mathcal{X} \rightarrow \mathcal{Y}$ is a compact operator, then C is a bounded operator.
- (b) If $C_1, C_2 : \mathcal{X} \rightarrow \mathcal{Y}$ are compact operators and $\alpha_1, \alpha_2 \in \mathbb{C}$, then $\alpha_1 C_1 + \alpha_2 C_2$ is compact.
- (c) If $C : \mathcal{X} \rightarrow \mathcal{Y}$ is a compact operator and $B_{\mathcal{X}} : \mathcal{Z} \rightarrow \mathcal{X}$ and $B_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Z}$ are bounded operators, then $C B_{\mathcal{X}}$ and $B_{\mathcal{Y}} C$ are compact.
- (d) Let, for each $i \in \mathbb{N}$, $C_i : \mathcal{X} \rightarrow \mathcal{Y}$ be a compact operator. If the C_i 's converge in operator norm to an operator $C : \mathcal{X} \rightarrow \mathcal{Y}$, then C is compact.

Proof: Let $\{x_i\}_{i \in \mathbb{N}}$ be a bounded sequence in \mathcal{X} .

(a) This is Problem 5, below.

(b) Since C_1 is compact, there is a subsequence $\{x_{i_\ell}\}_{\ell \in \mathbb{N}}$ such that $C_1 x_{i_\ell}$ converges in \mathcal{Y} . Since C_2 is compact, there is a subsequence $\{x_{i_{\ell_m}}\}_{m \in \mathbb{N}}$ of the bounded sequence $\{x_{i_\ell}\}_{\ell \in \mathbb{N}}$ such that $C_2 x_{i_{\ell_m}}$ converges in \mathcal{Y} . Then $\alpha_1 C_1 x_{i_{\ell_m}} + \alpha_2 C_2 x_{i_{\ell_m}}$ also converges in \mathcal{Y} .

(c) Let $\{z_i\}_{i \in \mathbb{N}}$ be a bounded sequence in \mathcal{Z} . Since $B_{\mathcal{X}}$ is bounded, $\{B_{\mathcal{X}} z_i\}_{i \in \mathbb{N}}$ is a bounded sequence in \mathcal{X} . Since C is compact, there is a subsequence $\{B_{\mathcal{X}} z_{i_\ell}\}_{\ell \in \mathbb{N}}$ such that $C B_{\mathcal{X}} z_{i_\ell}$ converges in \mathcal{Y} .

Since C is compact, there is a subsequence $\{x_{i_\ell}\}_{\ell \in \mathbb{N}}$ such that $C x_{i_\ell}$ converges in \mathcal{Y} . Since $C_{\mathcal{Y}}$ is bounded, $B_{\mathcal{Y}} C x_{i_\ell}$ converges in \mathcal{Y} .

(d) Let $\{x_j\}_{j \in \mathbb{N}}$ be a bounded sequence in \mathcal{X} and set

$$X = \sup_j \|x_j\|_{\mathcal{X}}$$

For each fixed $i \in \mathbb{N}$, $\{C_i x_j\}_{j \in \mathbb{N}}$ has a convergent subsequence, since C_i is compact by hypothesis. By taking subsequences of subsequences and using the diagonal trick, we can find a subsequence $\{x_{j_\ell}\}_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \rightarrow \infty} C_i x_{j_\ell}$ exists for all $i \in \mathbb{N}$. It suffices for us to prove that $\{C x_{j_\ell}\}_{\ell \in \mathbb{N}}$ is Cauchy. Let $\varepsilon > 0$. Since the C_i 's converge in operator norm to C , there is an $I \in \mathbb{N}$ such that $\|C - C_i\| < \frac{\varepsilon}{6X}$ for all $i \geq I$. Since $\{C_I x_{j_\ell}\}_{\ell \in \mathbb{N}}$ is Cauchy, there is an $L \in \mathbb{N}$ such that $\|C_I x_{j_\ell} - C_I x_{j_m}\|_{\mathcal{Y}} < \frac{\varepsilon}{3}$ for all $\ell, m > L$. Hence if $\ell, m > L$, then

$$\begin{aligned} \|C x_{j_\ell} - C x_{j_m}\|_{\mathcal{Y}} &\leq \|C x_{j_\ell} - C_I x_{j_\ell}\|_{\mathcal{Y}} + \|C_I x_{j_\ell} - C_I x_{j_m}\|_{\mathcal{Y}} + \|C_I x_{j_m} - C x_{j_m}\|_{\mathcal{Y}} \\ &\leq 2X\|C - C_I\| + \|C_I x_{j_\ell} - C_I x_{j_m}\|_{\mathcal{Y}} + 2X\|C_I - C\| \\ &< 2X\frac{\varepsilon}{6X} + \frac{\varepsilon}{3} + 2X\frac{\varepsilon}{6X} \\ &= \varepsilon \end{aligned}$$

■

Problem 5 Prove that compact operators are necessarily bounded.

Proposition 6 Let \mathcal{X} and \mathcal{Y} be Banach spaces. Denote by \mathcal{X}' and \mathcal{Y}' their dual spaces. That is, \mathcal{X}' (resp. \mathcal{Y}') is the Banach space of bounded linear functionals on \mathcal{X} (resp. \mathcal{Y}). The adjoint, $C^* : \mathcal{Y}' \rightarrow \mathcal{X}'$, of a bounded operator $C : \mathcal{X} \rightarrow \mathcal{Y}$ is determined by

$$(C^* \eta)(x) = \eta(Cx) \quad \text{for all } \eta \in \mathcal{Y}' \text{ and } x \in \mathcal{X}$$

A bounded operator $C : \mathcal{X} \rightarrow \mathcal{Y}$ is compact if and only if C^* is compact.

Proof: First assume that C is compact. Let $\{\eta_i\}_{i \in \mathbb{N}}$ be a bounded subset of \mathcal{Y}' and set

$$Y' = \sup_i \|\eta_i\|_{\mathcal{Y}'}$$

Let $B = \{x \in \mathcal{X} \mid \|x\|_{\mathcal{X}} \leq 1\}$ be the unit ball in \mathcal{X} . Since C is compact, \overline{CB} , which is the closure of $\{Cx \in \mathcal{Y} \mid \|x\|_{\mathcal{X}} \leq 1\}$, is a compact subset of \mathcal{Y} . We shall apply Arzelà–Ascoli to the sequence of functions

$$f_i : y \in \overline{CB} \mapsto \eta_i(y) \in \mathbb{C}$$

Since

$$|f_i(y)| \leq Y' \|y\|_{\mathcal{Y}} \leq Y' \|C\|$$

the sequence is uniformly bounded. Since

$$|f_i(y) - f_i(\tilde{y})| \leq Y' \|y - \tilde{y}\|_{\mathcal{Y}}$$

it is equicontinuous. So, by Arzelà–Ascoli, there is a subsequence f_{i_ℓ} that converges uniformly on \overline{CB} . Since

$$\begin{aligned} \|C^* \eta_i - C^* \eta_j\|_{\mathcal{X}'} &= \sup_{x \in B} |(C^* \eta_i)(x) - (C^* \eta_j)(x)| = \sup_{x \in B} |\eta_i(Cx) - \eta_j(Cx)| \\ &= \sup_{x \in B} |f_i(Cx) - f_j(Cx)| = \sup_{y \in CB} |f_i(y) - f_j(y)| \end{aligned}$$

the sequence $\{C^* \eta_{i_\ell}\}_{\ell \in \mathbb{N}}$ is Cauchy in \mathcal{X}' .

Conversely, assume that C^* is compact. Let $\{x_i\}_{i \in \mathbb{N}}$ be a bounded subset of \mathcal{X} . By the implication that we have already proven, $C^{**} : \mathcal{X}'' \rightarrow \mathcal{Y}''$ is compact. Even if \mathcal{X} and/or \mathcal{Y} is not reflexive, \mathcal{X} is a closed subspace of \mathcal{X}'' and \mathcal{Y} is a closed subspace of \mathcal{Y}'' . So we may view $\{x_i\}_{i \in \mathbb{N}}$ as a bounded subset of \mathcal{X}'' . Then $\{C^{**} x_i\}_{i \in \mathbb{N}}$ has a subsequence $\{C^{**} x_{i_\ell}\}_{\ell \in \mathbb{N}}$ that converges in \mathcal{Y}'' . For any $\eta \in \mathcal{Y}'$ and $x \in \mathcal{X}$ (we'll write X for x , when we want to think of it as an element of \mathcal{X}''),

$$\begin{aligned} (C^{**} X)(\eta) &= X(C^* \eta) && \text{by the definition of “adjoint”} \\ &= (C^* \eta)(x) && \text{by the identification of } \mathcal{X} \text{ with a subset of } \mathcal{X}'' \\ &= \eta(Cx) && \text{by the definition of “adjoint”} \end{aligned}$$

Thus $C^{**} x \in \mathcal{Y}''$ is $Cx \in \mathcal{Y}$, viewed as an element of \mathcal{Y}'' and $\{Cx_{i_\ell}\}_{\ell \in \mathbb{N}}$ converges in \mathcal{Y} . ■

It is the spectral properties of compact operators that make them act very much like matrices. Perhaps it is more appropriate to say that the spectral properties of noncompact operators are often very different from those of matrices. A simple, yet typical, example of this is given in Problem 6, below. We start with careful definitions of “eigenvalue” like terms. For a thorough, but still readable, treatment of the spectral theory of self-adjoint operators on Hilbert spaces, see[RS].

Definition 7 Let \mathcal{X} be a Banach space and $B : \mathcal{X} \rightarrow \mathcal{X}$ be a linear operator defined on \mathcal{X} .

(a) The number $\lambda \in \mathbb{C}$ is said to be in the resolvent set of B if the operator $\lambda \mathbb{1} - B$ is bijective (one-to-one and onto) with bounded inverse. We shall use $\rho(B)$ to denote the resolvent set of B .

(b) The number $\lambda \in \mathbb{C}$ is said to be in the spectrum of B if it is not in the resolvent set of B . We write $\sigma(B) = \mathbb{C} \setminus \rho(B)$.

(c) The number $\lambda \in \mathbb{C}$ is said to be an eigenvalue of B if there is a nonzero vector $x \in \mathcal{X}$, called an eigenvector corresponding to λ , such that $Bx = \lambda x$. The set of all eigenvalues of B is called the point spectrum of B .

Proposition 8 *Let \mathcal{X} be a Banach space and $B : \mathcal{X} \rightarrow \mathcal{X}$ be a linear operator defined on \mathcal{X} .*

(a) *If $|\lambda| > \|B\|$, then $\lambda \in \rho(B)$.*

(b) *$\rho(B)$ is an open subset of \mathbb{C} .*

(c) *If λ is an eigenvalue of B , then $\lambda \in \sigma(B)$.*

Proof: (a) Since $\frac{\|B\|}{|\lambda|} < 1$, the series $\frac{1}{\lambda} \sum_{m=0}^{\infty} \left(\frac{B}{\lambda}\right)^m$ converges in operator norm to a bounded operator R on \mathcal{X} . As

$$(\lambda \mathbb{1} - B)R = R(\lambda \mathbb{1} - B) = \sum_{m=0}^{\infty} \left(\frac{B}{\lambda}\right)^m - \sum_{m=0}^{\infty} \left(\frac{B}{\lambda}\right)^{m+1} = \mathbb{1}$$

$R = (B - \lambda \mathbb{1})^{-1}$ and $\lambda \in \rho(B)$.

(b) Let $\mu \in \rho(B)$ and denote by $(\mu \mathbb{1} - B)^{-1}$ the inverse of $\mu \mathbb{1} - B$. By hypothesis, this inverse is a bounded operator on \mathcal{X} . If $|\lambda - \mu| < \|(\mu \mathbb{1} - B)^{-1}\|$, then the series $(\mu \mathbb{1} - B)^{-1} \sum_{m=0}^{\infty} (\lambda - \mu)^m (\mu \mathbb{1} - B)^{-m}$ converges in operator norm to a bounded operator \tilde{R} on \mathcal{X} . As

$$\begin{aligned} (\lambda \mathbb{1} - B)\tilde{R} &= \tilde{R}(\lambda \mathbb{1} - B) = \tilde{R}(\mu \mathbb{1} - B) + (\lambda - \mu)\tilde{R} \\ &= \sum_{m=0}^{\infty} (\lambda - \mu)^m (\mu \mathbb{1} - B)^{-m} + \sum_{m=0}^{\infty} (\lambda - \mu)^{m+1} (\mu \mathbb{1} - B)^{-m-1} \\ &= \mathbb{1} \end{aligned}$$

\tilde{R} is the operator inverse of $(\lambda \mathbb{1} - B)$ and $\lambda \in \rho(B)$. This shows that

$$\{ \lambda \in \mathbb{C} \mid |\lambda - \mu| < \|(\mu \mathbb{1} - B)^{-1}\| \} \subset \rho(B)$$

and that $\rho(B)$ is open.

(c) If λ is an eigenvalue of B , then $\lambda \mathbb{1} - B$ has a nontrivial kernel, namely all of the eigenvectors corresponding to λ . Thus $\lambda \notin \rho(B)$. ■

The next problem shows that, for operators acting on infinite dimensional spaces, even nice operators, the bulk of the spectrum need not consist of eigenvalues.

Problem 6 Let $\mathcal{H} = L^2(X, \mu)$ for some measure space $\langle X, \mu \rangle$. Let $f : X \rightarrow \mathbb{C}$ be a bounded measurable function on X . Let A be the bounded linear operator on \mathcal{H} given by multiplication by $f(x)$.

(a) Prove that $\lambda \in \sigma(A)$ if and only if

$$\forall \epsilon > 0 \quad \mu\{x \in X \mid |f(x) - \lambda| < \epsilon\} > 0$$

(b) Prove that λ is an eigenvalue of A if and only if

$$\mu\{x \in X \mid f(x) = \lambda\} > 0$$

(c) Let X be the open interval $(0, 1)$, μ be Lebesgue measure on $(0, 1)$ and $f(x) = x$. Find the spectrum of A , the operator on \mathcal{H} given by multiplication by x . Also find all of the eigenvalues of A .

We next prove that if C is a compact operator, then $\sigma(C) \setminus \{0\}$ consists only eigenvalues of finite multiplicity. If there are infinitely many different eigenvalues, they must converge to zero. We first need the following technical lemma.

Lemma 9 *Let \mathcal{X} be a Banach space and $B : \mathcal{X} \rightarrow \mathcal{X}$ be a compact operator. If λ is a nonzero complex number, then the range of $\lambda\mathbb{1} - C$ is a closed linear subspace of \mathcal{X} .*

Proof: Denote by \mathcal{R} and \mathcal{K} the range and kernel, respectively, of $\lambda\mathbb{1} - C$. Let $y \in \overline{\mathcal{R}}$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{X} such that $(\lambda\mathbb{1} - C)x_n$ converges to y . Denote by ρ_n the distance from x_n to \mathcal{K} . For each $n \in \mathbb{N}$, there is a $z_n \in \mathcal{K}$ such that $\rho_n \leq \|x_n - z_n\| < \rho_n + \frac{1}{n}$. Then $\tilde{x}_n = x_n - z_n$ obeys

$$\lim_{n \rightarrow \infty} (\lambda\mathbb{1} - C)\tilde{x}_n = \lim_{n \rightarrow \infty} (\lambda\mathbb{1} - C)x_n = y$$

We first consider the case that $\{\rho_n\}_{n \in \mathbb{N}}$ is bounded. Then the sequence $\{\tilde{x}_n\}_{n \in \mathbb{N}}$ is bounded, and, since C is compact, there is a subsequence $\{\tilde{x}_{n_\ell}\}_{\ell \in \mathbb{N}}$ such that $C\tilde{x}_{n_\ell}$ converges in \mathcal{X} , say to $-\tilde{y}$. Then

$$\tilde{x}_{n_\ell} = \frac{1}{\lambda} [(\lambda\mathbb{1} - C)\tilde{x}_{n_\ell} + C\tilde{x}_{n_\ell}]$$

converges in \mathcal{X} to $x = \frac{1}{\lambda}(y - \tilde{y})$. Since C is bounded, $-\tilde{y} = Cx$ and $y = (\lambda\mathbb{1} - C)x \in \mathcal{R}$.

Finally, we consider the case that $\{\rho_n\}_{n \in \mathbb{N}}$ is not bounded. Then, possibly restricting to a subsequence, we may assume that $\lim_{n \rightarrow \infty} \rho_n = \infty$. As the sequence $\left\{ \frac{\tilde{x}_n}{\|\tilde{x}_n\|} \right\}_{n \in \mathbb{N}}$ is bounded and C is still compact, there is a subsequence $\left\{ \frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} \right\}_{\ell \in \mathbb{N}}$ such that $C \frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|}$ converges in \mathcal{X} , say to \tilde{z} . As

$$\lim_{n \rightarrow \infty} (\lambda \mathbb{1} - C) \frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} = \frac{y}{\lim_{n \rightarrow \infty} \|\tilde{x}_{n_\ell}\|} = 0$$

we have

$$\lim_{n \rightarrow \infty} \frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} = \frac{1}{\lambda} \lim_{n \rightarrow \infty} \left[(\lambda \mathbb{1} - C) \frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} + C \frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} \right] = \frac{\tilde{z}}{\lambda}$$

and hence

$$(\lambda \mathbb{1} - C) \tilde{z} = \lambda \lim_{n \rightarrow \infty} (\lambda \mathbb{1} - C) \frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|} = 0$$

In other words, $\tilde{z} \in \mathcal{K}$. This provides a contraction, since \tilde{x}_n is a distance ρ_n from \mathcal{K} so that $\frac{\tilde{x}_n}{\|\tilde{x}_n\|}$ is a distance $\frac{\rho_n}{\|\tilde{x}_n\|} \geq \frac{\rho_n}{\rho_n + 1/n}$ from \mathcal{K} . As $\lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_n + 1/n} = 1$, $\frac{\tilde{x}_{n_\ell}}{\|\tilde{x}_{n_\ell}\|}$ cannot converge to a point of \mathcal{K} . ■

Proposition 10 (The Fredholm Alternative) *Let $C : \mathcal{X} \rightarrow \mathcal{X}$ be a compact operator on the Banach space \mathcal{X} . If λ is a nonzero complex number, then either λ is an eigenvalue of C or $\lambda \in \rho(C)$.*

Proof: Suppose that λ is not an eigenvalue of C . Then, by definition, $\lambda \mathbb{1} - C$ is one-to-one. By lemma 9, the range of $\lambda \mathbb{1} - C$ is closed. We now claim that the range of $\lambda \mathbb{1} - C$ is all of \mathcal{X} . If not, $\mathcal{X}_1 = (\lambda \mathbb{1} - C)\mathcal{X}$ is a proper closed subspace of \mathcal{X} . Since the restriction of C to \mathcal{X}_1 is still compact, $\mathcal{X}_2 = (\lambda \mathbb{1} - C)\mathcal{X}_1$ is a closed subspace of \mathcal{X}_1 . If \mathcal{X}_2 were not a proper subspace of \mathcal{X}_1 , then for each $x \in \mathcal{X} \setminus \mathcal{X}_1$, there would be a vector $x' \in \mathcal{X}_1$ with $(\lambda \mathbb{1} - C)x' = (\lambda \mathbb{1} - C)x$ and this would contradict the assumption that $\lambda \mathbb{1} - C$ is one-to-one. Thus $\mathcal{X}_2 = (\lambda \mathbb{1} - C)\mathcal{X}_1$ is a proper closed subspace of \mathcal{X}_1 . Continuing in this way, we can generate a sequence $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$ of subspaces of \mathcal{X} with $\mathcal{X}_{n+1} = (\lambda \mathbb{1} - C)\mathcal{X}_n$ and \mathcal{X}_{n+1} a proper closed subspace of \mathcal{X}_n . By Problem 7, below, there is, for each $n \in \mathbb{N}$, a unit vector $x_n \in \mathcal{X}_n \setminus \mathcal{X}_{n+1}$ whose distance from \mathcal{X}_{n+1} is at least $\frac{1}{2}$. If $n > m$,

$$\frac{1}{\lambda}(Cx_m - Cx_n) = x_m - \tilde{x}_m$$

with

$$\tilde{x}_m = \frac{1}{\lambda}(\lambda \mathbb{1} - C)x_m + \frac{1}{\lambda}Cx_n \in \mathcal{X}_{m+1}$$

Hence $\|Cx_m - Cx_n\| \geq \frac{|\lambda|}{2}$ for all $n > m$ and $\{Cx_n\}_{n \in \mathbb{N}}$ may not contain any convergent subsequence, contradicting the compactness of C .

So $\lambda \mathbb{1} - C$ is both one-to-one and onto. The boundedness of the inverse map is an immediate consequence of the inverse mapping theorem [RS, Theorem III.11]. But it is

also easy to prove boundedness directly and we do that now. If $(\lambda\mathbb{1} - C)^{-1}$ is not bounded, there is a sequence of unit vectors $x_n \in \mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \|(\lambda\mathbb{1} - C)x_n\| = 0 \implies \lim_{n \rightarrow \infty} (\lambda\mathbb{1} - C)x_n = 0$$

Since C is compact, there is a subsequence $\{x_{n_m}\}_{m \in \mathbb{N}}$ such that Cx_{n_m} converges, say to y . But then

$$\lim_{m \rightarrow \infty} x_{n_m} = \lim_{m \rightarrow \infty} \frac{1}{\lambda} Cx_{n_m} + \lim_{m \rightarrow \infty} \frac{1}{\lambda} (\lambda\mathbb{1} - C)x_{n_m} = \frac{y}{\lambda}$$

and

$$Cy = \lambda C \lim_{m \rightarrow \infty} x_{n_m} = \lambda y$$

As $\|y\| = |\lambda| \neq 0$, this contradicts the assumption that λ is not an eigenvalue of C . ■

Problem 7 Let \mathcal{X} be a Banach space and \mathcal{Y} a proper closed subspace of \mathcal{X} . Let $0 < \rho < 1$. Prove that there is a unit vector $x \in \mathcal{X} \setminus \mathcal{Y}$ whose distance from \mathcal{Y} is at least ρ .

Problem 8 Let \mathcal{X} be an infinite dimensional Banach space. Prove that the identity operator on \mathcal{X} is not compact.

Proposition 11 (The Spectrum of Compact Operators) *Let $C : \mathcal{X} \rightarrow \mathcal{X}$ be a compact operator on the Banach space \mathcal{X} . The spectrum of C consists of at most countably many points. For any $\varepsilon > 0$, $\{ \lambda \in \sigma(C) \mid |\lambda| > \varepsilon \}$ is finite. If $0 \neq \lambda \in \sigma(C)$, then λ is an eigenvalue of C of finite multiplicity.*

Proof: We have already proven, in Proposition 10, that any nonzero number in the spectrum of C is an eigenvalue and we have also already proven, in Proposition 8, that $\sigma(C) \subset \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \|C\| \}$. Since eigenvectors corresponding to different eigenvalues are necessarily independent, it suffices to prove that there cannot exist a sequence $\{x_n\}_{n \in \mathbb{N}}$ of independent eigenvectors of C whose corresponding eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ converge to $\lambda \neq 0$.

Denote by \mathcal{X}_n the span of $\{x_1, x_2, \dots, x_n\}$. By Problem 7, there is, for each $n \geq 2$, a unit vector $y_n \in \mathcal{X}_n$ whose distance from \mathcal{X}_{n-1} is at least $\frac{1}{2}$. If $n > m$,

$$\frac{1}{\lambda_n} Cy_n - \frac{1}{\lambda_m} Cy_m = y_n - \tilde{y}_n$$

with

$$\tilde{y}_n = \frac{1}{\lambda_n} (\lambda_n \mathbb{1} - C)y_n + \frac{1}{\lambda_m} Cy_m \in \mathcal{X}_{n-1}$$

since $(\lambda_n \mathbb{1} - C)\mathcal{X}_n \subset \mathcal{X}_{n-1}$ and $C\mathcal{X}_m \subset \mathcal{X}_m \subset \mathcal{X}_{n-1}$. Hence $\|\frac{1}{\lambda_n} Cy_n - \frac{1}{\lambda_m} Cy_m\| \geq \frac{1}{2}$ for all $n > m$. By assumption $\lim_{n \rightarrow \infty} \lambda_n = \lambda \neq 0$, so that $\|Cy_n - Cy_m\| \geq \frac{|\lambda|}{4}$ for all $n > m$ sufficiently large. Thus $\{Cy_n\}_{n \in \mathbb{N}}$ may not contain any convergent subsequence, contradicting the compactness of C . ■

Problem 9 Let \mathcal{X} be an infinite dimensional Banach space and $C : \mathcal{X} \rightarrow \mathcal{X}$ a compact operator. Prove that $0 \in \sigma(C)$.

Problem 10 Let \mathcal{H} be a separable Hilbert space and let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for \mathcal{H} . Let $\{\mu_n\}_{n \in \mathbb{N}}$ be any sequence of complex numbers that converges to 0. Prove that the operator defined by

$$C\left(\sum_{n=1}^{\infty} \alpha_n e_n\right) = \sum_{n=1}^{\infty} \mu_n \alpha_n e_{n+1}$$

is compact and has $\sigma(C) = \{0\}$.

References

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