Analytic Banach Space Valued Functions

Let $\mathcal{B}$ be a Banach space and $\mathcal{D}$ be an open subset of $\mathbb{C}$.

Definition 1 (Analytic) Let $\mathbf{f}: \mathcal{D} \to \mathcal{B}$.

(a) $\mathbf{f}$ is analytic at $z_0 \in \mathcal{D}$ if

$$\mathbf{f}'(z_0) = \lim_{z \to z_0} \tfrac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0}$$

exists (in the norm on $\mathcal{B}$).

(b) $\mathbf{f}$ is weakly analytic at $z_0 \in \mathcal{D}$ if

$$\lim_{z \to z_0} \tfrac{\ell(\mathbf{f}(z))-\ell(\mathbf{f}(z_0))}{z-z_0}$$

exists for each $\ell \in \mathcal{B}^*$. In other words, $\mathbf{f}$ is weakly analytic at $z_0 \in \mathcal{D}$ if, for each $\ell \in \mathcal{B}^*$, the function $F: \mathcal{D} \to \mathbb{C}$ defined by $F(z) = \ell\big(\mathbf{f}(z)\big)$ is analytic at z_0.

Theorem 2 *Let* $\mathbf{f}: \mathcal{D} \to \mathcal{B}$.

(a) $\mathbf{f}(z)$ *is analytic on* $\mathcal{D}$ *if and only if* $\mathbf{f}(z)$ *is weakly analytic on* $\mathcal{D}$.

(b) If $\mathbf{f}(z)$ *is analytic on* $\mathcal{D}$ *and* $\mathcal{K} \subset \mathcal{D}$ *is compact, then* $\|\mathbf{f}(z)\|$ *is bounded on* $\mathcal{K}$.

(c) If $\mathbf{f}(z)$ *is analytic on* $\mathcal{D}$*, then* $\mathbf{f}'(z)$ *is analytic on* $\mathcal{D}$.

(d) If $\mathbf{f}(z)$ *is analytic on* $\mathcal{D}$*, then* $\mathbf{f}(z)$ *is continuous on* $\mathcal{D}$.

(e) If $\mathbf{f}(z)$ *and* $\mathbf{g}(z)$ *are analytic on* $\mathcal{D}$ *and* $\alpha, \beta \in \mathbb{C}$*, then* $\alpha\mathbf{f}(z) + \beta\mathbf{g}(z)$ *is analytic on* $\mathcal{D}$.

(f) Let, for each $n \in \mathbb{N}$, $\mathbf{f}_n : \mathcal{D} \to \mathcal{B}$ *be analytic on* $\mathcal{D}$*. If the sequence* $\big\{\mathbf{f}_n\big\}_{n\in\mathbb{N}}$ *is uniformly bounded on* $\mathcal{D}$ *and converges weakly to* $\mathbf{f}(z)$*, then* $\mathbf{f}(z)$ *is analytic on* $\mathcal{D}$.

(g) Let $z \in \mathcal{D}$, $\mathbf{f}$ *be analytic on* $\mathcal{D}$ *and* Γ *be a simple, closed, continuous, piecewise* C^2 *curve in* $\mathcal{D}$ *that is positively oriented and contains* z *in its interior. Then*

$$\mathbf{f}(z) = \tfrac{1}{2\pi i} \int_\Gamma \tfrac{\mathbf{f}(\zeta)}{\zeta - z}\, d\zeta$$

Here the integral is defined as the norm limit of its Riemann partial sums.

(h) Let $z_0 \in \mathcal{D}$*. If* $\mathbf{f}(z)$ *is analytic on the open ball,* $B_r(z_0)$*, of radius* r *centred on* z_0*, then* $\mathbf{f}(z)$ *has a unique power series expansion* $f(z) = \sum_{n=0}^{\infty} \mathbf{A}_n \,(z-z_0)^n$ *on* $B_r(z_0)$*. This series is norm convergent on* $B_r(z_0)$.

(i) Let $z_0 \in \mathcal{D}$ and $0 \le R_1 < R_2$. If $\mathbf{f}(z)$ is analytic on the annulus, $R_1 < |z - z_0| < R_2$, then $f(z)$ has a unique Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} A_n\,(z - z_0)^n$ on $R_1 < |z_0| < R_2$. This series is norm convergent on $R_1 < |z - z_0| < R_2$.

Proof of Theorem 2.a:

$\Rightarrow$: This is easy since, for each $z_0 \in \mathcal{D}$ and $\ell \in \mathcal{B}^*$,

$$\left|\tfrac{\ell(\mathbf{f}(z))-\ell(\mathbf{f}(z_0))}{z-z_0} - \ell\big(\mathbf{f}'(z_0)\big)\right| = \left|\ell\Big(\tfrac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0} - \mathbf{f}'(z_0)\Big)\right| \le \|\ell\|\ \left\|\tfrac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0} - \mathbf{f}'(z_0)\right\|$$

$\Leftarrow$: Let $z_0 \in \mathcal{D}$. We must show that $\mathbf{f}(z)$ is analytic at $\mathbf{z}_0$. Let Γ be a positively oriented circle that is contained in $\mathcal{D}$ and is centred on z_0. Denote by r the radius of Γ. Then, for each $\ell \in \mathcal{B}^*$, $F_\ell(z) = \ell\big(\mathbf{f}(z)\big)$ is a analytic function of z, so that, for each z, z' in the interior of Γ,

$$\begin{aligned}
\ell\Big(\tfrac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0} - \tfrac{\mathbf{f}(z')-\mathbf{f}(z_0)}{z'-z_0}\Big) &= \tfrac{1}{2\pi i}\int_\Gamma \big[\tfrac{1}{z-z_0}\big(\tfrac{1}{\zeta-z} - \tfrac{1}{\zeta-z_0}\big) - \tfrac{1}{z'-z_0}\big(\tfrac{1}{\zeta-z'} - \tfrac{1}{\zeta-z_0}\big)\big]\ell\big(\mathbf{f}(\zeta)\big)\ d\zeta \\
&= \tfrac{1}{2\pi i}\int_\Gamma \big[\tfrac{1}{(\zeta-z)(\zeta-z_0)} - \tfrac{1}{(\zeta-z')(\zeta-z_0)}\big]\ell\big(\mathbf{f}(\zeta)\big)\ d\zeta \\
&= \tfrac{1}{2\pi i}\int_\Gamma \tfrac{z-z'}{(\zeta-z)(\zeta-z')(\zeta-z_0)}\ell\big(\mathbf{f}(\zeta)\big)\ d\zeta
\end{aligned}$$

by the Cauchy integral formula, applied to $F_\ell(z)$, $F_\ell(z_0)$ and $F_\ell(z')$.

Think of $\mathbf{f}(\zeta)$ as the element of $\big(\mathcal{B}^*\big)^*$ that maps $\ell \in \mathcal{B}^*$ to $\ell\big(\mathbf{f}(\zeta)\big)$. Since $F_\ell(\zeta)$ is continuous in ζ and Γ is compact, there is a constant C_ℓ such that $\big|\ell\big(\mathbf{f}(\zeta)\big)\big| \le C_\ell$ for all $\zeta \in \Gamma$. Thus the set of linear operators $\mathcal{F} = \big\{\ \mathbf{f}(\zeta)\ \big|\ \zeta \in \Gamma\ \big\} \subset \big(\mathcal{B}^*\big)^*$ is pointwise bounded. By the principle of uniform boundedness we have that

$$\sup_{\zeta\in\Gamma} \|\mathbf{f}(\zeta)\|_{(\mathcal{B}^*)^*} \le C < \infty \implies \sup_{\zeta\in\Gamma} \big|\ell\big(\mathbf{f}(\zeta)\big)\big| \le C\|\ell\|_{\mathcal{B}^*}$$

Hence if $|z - z_0|, |z' - z_0| \le \frac{r}{2}$,

$$\left|\ell\Big(\tfrac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0} - \tfrac{\mathbf{f}(z')-\mathbf{f}(z_0)}{z'-z_0}\Big)\right| \le \tfrac{1}{2\pi}\ \tfrac{|z-z'|}{\frac{r}{2}\times\frac{r}{2}\times r}\ C\|\ell\|_{\mathcal{B}^*}\ 2\pi r = \tfrac{4C}{r^2}|z - z'|\ \|\ell\|_{\mathcal{B}^*}$$

so that

$$\left\|\tfrac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0} - \tfrac{\mathbf{f}(z')-\mathbf{f}(z_0)}{z'-z_0}\right\|_{\mathcal{B}} \le \tfrac{4C}{r^2}|z - z'|$$

converges to 0 as $z, z' \to z_0$. Since $\mathcal{B}$ is complete, $\frac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0}$ converges to some $\mathbf{f}'(z_0) \in \mathcal{B}$ as $z \to z_0$. ■

Remark 3 In the case that $\mathbf{f} : \mathcal{D} \to \mathcal{B} = \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$, for some Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$, the above proof of Theorem 2.a shows that

$$\mathbf{f}(z) \textit{ is analytic on } \mathcal{D} \iff \ell\big(\mathbf{f}(z)\mathbf{x}\big) \textit{ is analytic on } \mathcal{D} \textit{ for each } \mathbf{x} \in \mathcal{B}_1,\ \ell \in \mathcal{B}_2^*$$

Proof of Theorem 2.b:

$$\begin{aligned}
\mathbf{f}(z) \text{ analytic on } \mathcal{D} &\implies \ell\big(\mathbf{f}(z)\big) \text{ analytic on } \mathcal{D} \text{ for each } \ell \in \mathcal{B}^* \\
&\implies \ell\big(\mathbf{f}(z)\big) \text{ continuous on } \mathcal{D} \text{ for each } \ell \in \mathcal{B}^* \\
&\implies \ell\big(\mathbf{f}(z)\big) \text{ bounded on } \mathcal{K} \text{ for each } \ell \in \mathcal{B}^* \\
&\implies \|\mathbf{f}(z)\| \text{ bounded on } \mathcal{K} \text{ by the principle of uniform boundedness}
\end{aligned}$$

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Proof of Theorem 2.c:

$$\begin{aligned}
\mathbf{f}(z) \text{ analytic on } \mathcal{D} &\implies \mathbf{f}(z) \text{ weakly analytic on } \mathcal{D} \\
&\implies \mathbf{f}'(z) \text{ weakly analytic on } \mathcal{D}\text{, by ordinary complex analysis} \\
&\implies \mathbf{f}'(z) \text{ analytic on } \mathcal{D}
\end{aligned}$$

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Proof of Theorem 2.d: Let $z_0 \in \mathcal{D}$. Since $\mathbf{f}(z)$ is analytic at z_0, $\frac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0}$ converges to $\mathbf{f}'(z_0) \in \mathcal{B}$ as $z \to z_0$. Hence there are $\varepsilon > 0$ and $C < \infty$ such that $\big\|\frac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0}\big\| \le C$ for all $|z - z_0| < \varepsilon$. Consequently

$$\|\mathbf{f}(z) - \mathbf{f}(z_0)\| \le |z - z_0|\ \big\|\tfrac{\mathbf{f}(z)-\mathbf{f}(z_0)}{z-z_0}\big\| \le C|z - z_0|$$

for all $|z - z_0| < \varepsilon$. ■

Proof of Theorem 2.e:

$$\begin{aligned}
\mathbf{f}(z), \mathbf{g}(z) \text{ analytic on } \mathcal{D} &\implies \mathbf{f}(z), \mathbf{g}(z) \text{ weakly analytic on } \mathcal{D} \\
&\implies \alpha\mathbf{f}(z) + \beta\mathbf{g}(z) \text{ weakly analytic on } \mathcal{D} \\
&\implies \alpha\mathbf{f}(z) + \beta\mathbf{g}(z) \text{ analytic on } \mathcal{D}
\end{aligned}$$

■

Remark 4 In the case that $\mathbf{f} : \mathcal{D} \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ and $\mathbf{g} : \mathcal{D} \to \mathcal{L}(\mathcal{B}_2, \mathcal{B}_3)$, for some Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathbf{f}(z), \mathbf{g}(z)$ are analytic on $\mathcal{D}$, then $\mathbf{f}(z)\mathbf{g}(z)$ is also analytic on $\mathcal{D}$ since, for each $z_0 \in \mathcal{D}$,

$$\begin{aligned}
&\tfrac{1}{z-z_0}\big[\mathbf{f}(z)\mathbf{g}(z) - \mathbf{f}(z_0)\mathbf{g}(z_0)\big] = \mathbf{f}(z)\tfrac{1}{z-z_0}\big[\mathbf{g}(z) - \mathbf{g}(z_0)\big] + \tfrac{1}{z-z_0}\big[\mathbf{f}(z) - \mathbf{f}(z_0)\big]\mathbf{g}(z_0) \\
&= \mathbf{f}(z_0)\tfrac{1}{z-z_0}\big[\mathbf{g}(z) - \mathbf{g}(z_0)\big] + \big[f(z) - f(z_0)\big]\tfrac{1}{z-z_0}\big[\mathbf{g}(z) - \mathbf{g}(z_0)\big] + \tfrac{1}{z-z_0}\big[\mathbf{f}(z) - \mathbf{f}(z_0)\big]\mathbf{g}(z_0)
\end{aligned}$$

converges in operator norm to $\mathbf{f}(z_0)\mathbf{g}'(z_0) + \mathbf{f}'(z_0)\mathbf{g}(z_0)$ as $z \to z_0$, by the definitions of $\mathbf{f}'(z_0)$ and $\mathbf{g}'(z_0)$ and by Theorem 2.b,d.

Proof of Theorem 2.f: Set $C = \sup_{z\in\mathcal{D}, n\in\mathbb{N}} \|\mathbf{f}_n(z)\|_{\mathcal{B}} < \infty$. Then for each $\ell \in \mathcal{B}^*$ we have

- ∘ $\ell\big(\mathbf{f}_n(z)\big)$ is analytic on $\mathcal{D}$ for all $n \in \mathbb{N}$ and
- ∘ $\big|\ell\big(\mathbf{f}_n(z)\big)\big| \le C\|\ell\|_{cB^*}$ for all $z \in \mathcal{D}$ and $n \in \mathbb{N}$

so that

$$\begin{aligned}
\mathbf{f}_n(z) \to \mathbf{f}(z) \text{ weakly on } \mathcal{D} &\implies \ell\big(\mathbf{f}_n(z)\big) \to \ell\big(\mathbf{f}(z)\big) \text{ on } \mathcal{D} \\
&\implies \ell\big(\mathbf{f}(z)\big) \text{ analytic on } \mathcal{D}\text{, by ordinary complex analysis}
\end{aligned}$$

Thus $\mathbf{f}(z)$ is weakly analytic, and hence analytic, on $\mathcal{D}$. ■

Proof of Theorem 2.g: For simplicity, we just consider the case that Γ is C^2. Let $\zeta = \gamma(t)$, $0 \le t \le 1$ be a, properly oriented, C^1 parametrization of Γ. Set $\mathbf{g}(t) = \frac{\mathbf{f}(\gamma(t))}{\gamma(t)-z}\gamma'(t)$. By parts (b) and (c), $G = \sup_{0<t<1} \big\|\frac{d}{dt}\mathbf{g}(t)\big\| < \infty$. By Problem 1, below, for any $0 \le t, t' \le 1$,

$$\|\mathbf{g}(t) - \mathbf{g}(t')\| \le G|t - t'|$$

and we can define

$$\tfrac{1}{2\pi i}\int_\Gamma \tfrac{\mathbf{f}(\zeta)}{\zeta - z}\, d\zeta = \tfrac{1}{2\pi i}\int_0^1 \mathbf{g}(t)\, dt = \tfrac{1}{2\pi i}\lim_{\mathbb{P},\mathbb{T}} R\big(\mathbb{P}, \mathbb{T}, \mathbf{g}(t)\big)$$

as a (norm) limit of Riemann partial sums in the usual way. See Problem 2, below.

For any $\ell \in \mathcal{B}^*$, the function $z \in \mathcal{D} \mapsto \ell\big(\mathbf{f}(z)\big) \in \mathbb{C}$ is analytic on $\mathcal{D}$ so that

$$\begin{aligned}
\ell\big(\mathbf{f}(z)\big) &= \tfrac{1}{2\pi i}\int_\Gamma \tfrac{\ell(\mathbf{f}(\zeta))}{\zeta - z}\, d\zeta &&= \tfrac{1}{2\pi i}\lim_{\mathbb{P},\mathbb{T}} R\Big(\mathbb{P}, \mathbb{T}, \ell\big(\mathbf{g}(t)\big)\Big) \\
&= \tfrac{1}{2\pi i}\ell\Big(\lim_{\mathbb{P},\mathbb{T}} R\big(\mathbb{P}, \mathbb{T}, \mathbf{g}(t)\big)\Big) &&= \tfrac{1}{2\pi i}\ell\Big(\int_\Gamma \tfrac{\mathbf{f}(\zeta)}{\zeta - z}\, d\zeta\Big)
\end{aligned}$$

■

Problem 1 Let $\mathbf{g} : [0,1] \to \mathcal{B}$ be differentiable and obey

$$G = \sup_{0<t<1} \left\| \tfrac{d}{dt}\mathbf{g}(t) \right\| < \infty$$

Prove that

$$\|\mathbf{g}(t) - \mathbf{g}(t')\| \le G|t - t'|$$

for all $0 \le t, t' \le 1$.

Problem 2 Let $\mathbf{g} : [0,1] \to \mathcal{B}$ obey

$$\|\mathbf{g}(t) - \mathbf{g}(t')\| \le G|t - t'|$$

for all $0 \le t, t' \le 1$. Define

- a partition of $[0,1]$ to be a finite set of points

$$\mathbb{P} = \big\{t_0, t_1, \cdots, t_n\big\} \qquad \text{with} \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

- a choice for the partition $\mathbb{P}$ to be a finite set of points

$$\mathbb{T} = \big\{t_1^*, \cdots, t_n^*\big\} \qquad \text{with} \quad t_{j-1} \le t_j^* \le t_j \text{ for all } 1 \le j \le n$$

- the Riemann partial sum for partition $\mathbb{P}$, choice $\mathbb{T}$ and function $\mathbf{g}$

$$R\big(\mathbb{P}, \mathbb{T}, \mathbf{g}(t)\big) = \sum_{j=1}^{n} \mathbf{g}(t_j^*)[t_j - t_{j-1}]$$

- the function $\mathbf{g}(t)$ to be Riemann integrable on $[0,1]$ if there is an $\mathbf{I} \in \mathcal{B}$ such that

$$\forall\, \varepsilon > 0\ \exists\ \mathbb{P}_\varepsilon \text{ such that } \mathbb{P}_\varepsilon \subset \mathbb{P} \implies \big|R\big(\mathbb{P}, \mathbb{T}, \mathbf{g}(t)\big) - \mathbf{I}\big| < \varepsilon$$

  This is denoted

$$\int_0^1 \mathbf{g}(t)\ dt = \lim_{\mathbb{P},\mathbb{T}} R\big(\mathbb{P}, \mathbb{T}, \mathbf{g}(t)\big)$$

(a) Prove that for any partitions $\mathbb{P}, \mathbb{P}'$ with $\mathbb{P} \subset \mathbb{P}'$ and any choices $\mathbb{T}, \mathbb{T}'$ for $\mathbb{T}$ and $\mathbb{T}'$, respectively,

$$\big\|R\big(\mathbb{P}, \mathbb{T}, \mathbf{g}(t)\big) - R\big(\mathbb{P}', \mathbb{T}', \mathbf{g}(t)\big)\big\| \le G\, M(\mathbb{P})$$

where the "mesh"

$$M(\mathbb{P}) = \max_{1 \le j \le n} [t_j - t_{j-1}]$$

(b) Prove that $\mathbf{g}(t)$ is Riemann integrable on $[0,1]$.

Proof of Theorem 2.h: The proof is the same as in ordinary complex analysis. Let $0 < r' < r$ and denote by Γ the positively oriented circle of radius r' centred on z_0. For any $z \in B_{r'}(z_0)$, expand

$$\begin{aligned}\mathbf{f}(z) &= \tfrac{1}{2\pi i}\int_\Gamma \tfrac{\mathbf{f}(\zeta)}{\zeta - z}\,d\zeta = \tfrac{1}{2\pi i}\int_\Gamma \tfrac{\mathbf{f}(\zeta)}{\zeta - z_0}\big[1 - \tfrac{z-z_0}{\zeta-z_0}\big]^{-1}\,d\zeta \\ &= \tfrac{1}{2\pi i}\int_\Gamma \sum_{n=0}^\infty \tfrac{\mathbf{f}(\zeta)}{\zeta-z_0}\big(\tfrac{z-z_0}{\zeta-z_0}\big)^n\,d\zeta\end{aligned}$$

Since $\big|\tfrac{z-z_0}{\zeta-z_0}\big| \le \tfrac{|z-z_0|}{r'} < 1$, the sum over n converges in norm, uniformly in ζ, and we may pull it outside the integral and define

$$\mathbf{A}_n = \tfrac{1}{2\pi i}\int_\Gamma \tfrac{\mathbf{f}(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta$$

■

Proof of Theorem 2.i: The proof is again the same as the proof in ordinary complex analysis. Write

$$\mathbf{f}(z) = \tfrac{1}{2\pi i}\int_{\Gamma_{r_2}} \tfrac{\mathbf{f}(\zeta)}{\zeta-z}\,d\zeta - \tfrac{1}{2\pi i}\int_{\Gamma_{r_1}} \tfrac{\mathbf{f}(\zeta)}{\zeta-z}\,d\zeta$$

where Γ_r is the positively oriented circle of radius r centred on z_0 and where $0 < R_1 < r_1 < |z - z_0| < r_2 < R_2$. Then expand

$$\tfrac{1}{\zeta-z} = \tfrac{1}{\zeta-z_0}\big[1 - \tfrac{z-z_0}{\zeta-z_0}\big]^{-1} = \sum_{n=0}^\infty \tfrac{1}{\zeta-z_0}\big(\tfrac{z-z_0}{\zeta-z_0}\big)^n$$

in the first integral and

$$-\tfrac{1}{\zeta-z} = \tfrac{1}{z-z_0}\big[1 - \tfrac{\zeta-z_0}{z-z_0}\big]^{-1} = \sum_{n=0}^\infty \tfrac{1}{z-z_0}\big(\tfrac{\zeta-z_0}{z-z_0}\big)^n$$

in the second integral. See, for example, [Lars Ahlfors, *Complex Analysis*, §5.1.3] or [John B. Conway, *Functions of One Complex Variable I*, §V.1.11].

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Remark 5 More generally, one can fairly simply use the above ideas to define and exploit analyticity for functions defined on a Banach space and taking values in another Banach space. See [Jürgen Pöschel and Eugene Trubowitz, *Inverse Spectral Theory*, Appendix A].