Euler Wobble

Consider a rigid body with inertia tensor

\[ \mathcal{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad \text{with} \quad I_1 = 1, \ I_2 = \frac{1}{2}, \ I_3 = \frac{1}{3} \]

Under free motion, the angular momentum \( \vec{\mathcal{M}} \), expressed in body coordinates, obeys Euler’s equations (see “A Summary of Rigid Body Formulae”)

\[ \dot{M}_1 = M_2 M_3 \quad \dot{M}_2 = -2M_1 M_3 \quad \dot{M}_3 = M_1 M_2 \]

These equations imply that both \( \vec{\mathcal{M}}^2 = M_1^2 + M_2^2 + M_3^2 \) and the energy \( E = M_1^2 + 2M_2^2 + 3M_3^2 \) are conserved:

\[
\frac{d}{dt} (M_1^2 + M_2^2 + M_3^2) = 2M_1 \dot{M}_1 + 2M_2 \dot{M}_2 + 2M_3 \dot{M}_3 = 2M_1 M_2 M_3 - 4M_1 M_2 M_3 + 2M_1 M_2 M_3 = 0 \\
\frac{d}{dt} (M_1^2 + 2M_2^2 + 3M_3^2) = 2M_1 \dot{M}_1 + 4M_2 \dot{M}_2 + 6M_3 \dot{M}_3 = 2M_1 M_2 M_3 - 8M_1 M_2 M_3 + 6M_1 M_2 M_3 = 0
\]

Assuming that \( \vec{\mathcal{M}}^2 \neq 0 \), we may choose units of time so that

\[ M_1^2 + M_2^2 + M_3^2 = 1 \quad M_1^2 + 2M_2^2 + 3M_3^2 = E \quad (\ast) \]

Observe that the condition \( M_1^2 + M_2^2 + M_3^2 = 1 \) forces

\[ 1 = M_1^2 + M_2^2 + M_3^2 \leq M_1^2 + 2M_2^2 + 3M_3^2 \leq 3M_1^2 + 3M_2^2 + 3M_3^2 = 3 \quad \text{so that} \quad 1 \leq E \leq 3 \]

Here are sketches of the curve \( (\ast) \) for various values of \( E \).

**Case 1:** \( E = 1 \). In this case, \( (\ast) \) forces \( \vec{\mathcal{M}} = \pm(1, 0, 0) \).

**Case 2:** \( 1 < E < 2 \). The equations \( (\ast) \) are equivalent to \( M_1^2 + M_2^2 + M_3^2 = 1, \ M_1^2 + 2M_2^2 = E - 1 \). So \( (\ast) \) is the intersection of the unit sphere with an elliptical cylinder centred on the \( x \)-axis.

**Case 3:** \( E = 2 \). The equations \( (\ast) \) are equivalent to \( M_1^2 + M_2^2 + M_3^2 = 1, \ M_1^2 - M_3^2 = 0 \). So \( (\ast) \) is the intersection of the unit sphere with the planes \( M_1 = \pm M_3 \). Each sign gives a great circle.

**Case 4:** \( 2 < E < 3 \). The equations \( (\ast) \) are equivalent to \( M_1^2 + M_2^2 + M_3^2 = 1, \ 2M_1^2 + M_2^2 = 3 - E \). So \( (\ast) \) is the intersection of the unit sphere with an elliptical cylinder centred on the \( z \)-axis.

**Case 5:** \( E = 3 \). In this case, \( (\ast) \) forces \( \vec{\mathcal{M}} = \pm(0, 0, 1) \).
The following figure provides a sketch of some representative trajectories in the first octant.

We conclude that
- Trajectories that start exactly at \((\pm 1, 0, 0)\) or \((0, \pm 1, 0)\) or \((0, 0, \pm 1)\) (i.e. rotations of the rigid body exactly about one of its principal axes) do not move at all. That is, \((\pm 1, 0, 0)\), \((0, \pm 1, 0)\) and \((0, 0, \pm 1)\) are critical points.
- Trajectories that start near, but not exactly at \((\pm 1, 0, 0)\) or \((0, 0, \pm 1)\) (i.e. rotations of the rigid body almost about the longest or shortest principal axes) just circle about their starting points for all time. That is, \((\pm 1, 0, 0)\) and \((0, 0, \pm 1)\) are stable critical points.
- Trajectories that start near, but not exactly at \((0, 1, 0)\) (i.e. rotations of the rigid body almost about the middle principal axis) move to near \((0, -1, 0)\) and then back to near \((0, 1, 0)\) and so on. There is one exceptional trajectory (called the separatrix) that starts near \((0, 1, 0)\) and tries to go to \((0, -1, 0)\) exactly, but slows as it gets closer and closer to \((0, -1, 0)\) and never actually gets to \((0, -1, 0)\). So \((0, \pm 1, 0)\) are unstable critical points.