Torus Geodesics

Let $0 < \rho < R$ be constants. The surface in $\mathbb{R}^3$ whose equation in cylindrical coordinates is

$$(r - R)^2 + z^2 = \rho^2$$

is a torus, which we shall call $M$. Use as coordinates on $M$ two angles $\theta$ and $\varphi$ determined by

$$x = (R + \rho \cos \varphi) \cos \theta \quad y = (R + \rho \cos \varphi) \sin \theta \quad z = \rho \sin \varphi$$

By way of a check, observe that $r = R + \rho \cos \varphi$ and $z = \rho \sin \varphi$ obey the equation $(r - R)^2 + z^2 = \rho^2$. For any curve on $M$

$$\dot{x}(t) = -\dot{\varphi}(t) \rho \sin \varphi(t) \cos \theta(t) - \dot{\theta}(t) (R + \rho \cos \varphi(t)) \sin \theta(t)$$
$$\dot{y}(t) = -\dot{\varphi}(t) \rho \sin \varphi(t) \sin \theta(t) + \dot{\theta}(t) (R + \rho \cos \varphi(t)) \cos \theta(t)$$
$$\dot{z}(t) = \dot{\varphi}(t) \rho \cos \varphi(t)$$

Thus

$$\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2 = \rho^2 \dot{\varphi}(t)^2 + (R + \rho \cos \varphi(t))^2 \dot{\theta}(t)^2$$

and the Lagrangian for free motion on $M$ is $L(\theta, \varphi, v_\theta, v_\varphi) = \frac{1}{2} \rho^2 v_\varphi^2 + \frac{1}{2} (R + \rho \cos \varphi)^2 v_\theta^2$. The $\theta$ Euler–Lagrange equation is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} (\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \right) = \frac{\partial L}{\partial \theta} (\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t))$$

$$\implies \frac{d}{dt} \left( (R + \rho \cos \varphi(t))^2 \dot{\theta}(t) \right) = 0$$

$$\implies (R + \rho \cos \varphi(t))^2 \ddot{\theta}(t) = p_\theta, \text{constant} \quad (EL_\theta)$$

The $\varphi$ Euler–Lagrange equation is

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} (\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \right) = \frac{\partial L}{\partial \varphi} (\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t))$$

$$\implies \rho^2 \ddot{\varphi}(t) = -\rho (R + \rho \cos \varphi(t)) \sin \varphi(t) \dot{\theta}(t)^2$$

$$\implies \rho^2 \ddot{\varphi}(t) = -\rho (R + \rho \cos \varphi(t))^{-3} \sin \varphi(t) \rho_\varphi^2 \quad (EL_\varphi)$$
By conservation of energy

\[
\frac{1}{2} \rho^2 \dot{\varphi}(t)^2 + \frac{1}{2} (R + \rho \cos \varphi(t))^2 \dot{\theta}(t)^2 = E, \text{ constant}
\]

\[
\Rightarrow \frac{1}{2} \rho^2 \dot{\varphi}(t)^2 = E - \frac{1}{2} (R + \rho \cos \varphi(t))^{-2} p_\theta^2
\] (E)

Observe that, by making appropriate choices of initial conditions, we may achieve any value of \( p_\theta \in \mathbb{R} \). (For example, we could choose \( \theta(0) = \varphi(0) = 0, \dot{\varphi}(0) = 0 \) and \( \dot{\theta}(0) = \frac{p_\theta}{(R + \rho)^2} \).) For any fixed \( p_\theta \), \( E = \frac{1}{2} \rho^2 \dot{\varphi}(t)^2 + \frac{1}{2} (R + \rho \cos \varphi(t))^{-2} p_\theta^2 \geq \frac{1}{2} (R + \rho)^{-2} p_\theta^2 \) since \( \cos \varphi(t) \leq 1 \). By making appropriate choices of initial conditions, we may achieve any value of \( E \geq \frac{1}{2} (R + \rho)^{-2} p_\theta^2 \). Observe that \( \frac{1}{2} \rho^2 \dot{\varphi}(t)^2 + \frac{1}{2} (R + \rho \cos \varphi(t))^{-2} p_\theta^2 \) is exactly the sum of the kinetic and potential energies for a particle of mass \( \rho^2 \) moving in one dimension with potential energy \( V(\varphi) = \frac{1}{2} (R + \rho \cos \varphi)^{-2} p_\theta^2 \).

So you can develop some intuition about the behaviour of \( \varphi(t) \) by imagining what happens to a particle moving on the surface in the figure above.

- If \( p_\theta = 0 \), then, from (EL\( \theta \)) and (EL\( \varphi \)), \( \dot{\theta}(t) = \ddot{\varphi}(t) = 0 \) for all \( t \) and the constant speed geodesic sweeps out a circle with \( \theta \) and \( \dot{\varphi} \) constant. In the figure on the left below, the heavy line is the top half of the geodesic.

- If \( p_\theta \neq 0 \) and \( E = \frac{1}{2} (R + \rho)^{-2} p_\theta^2 \), then the condition

\[
\frac{1}{2} (R + \rho \cos \varphi(t))^{-2} p_\theta^2 \leq E = \frac{1}{2} (R + \rho)^{-2} p_\theta^2 \iff (R + \rho \cos \varphi(t))^{-2} \leq (R + \rho)^{-2}
\]

forces \( \cos \varphi(t) \geq 1 \) and hence \( \varphi(t) = 0 \) for all \( t \). The geodesic sweeps out the outside equator of the torus, \( r = R + \rho, z = 0 \), with \( \dot{\theta} \) constant. In the figure on the right above, the heavy line is the one quarter of the geodesic.

- If \( p_\theta \neq 0 \) and \( \frac{1}{2} (R + \rho)^{-2} p_\theta^2 < E < \frac{1}{2} (R - \rho)^{-2} p_\theta^2 \), then the condition

\[
\frac{1}{2} (R + \rho \cos \varphi(t))^{-2} p_\theta^2 \leq E \iff (R + \rho \cos \varphi(t))^{-2} \leq \frac{2E}{p_\theta^2} \iff R + \rho \cos \varphi(t) \geq \frac{p_\theta}{\sqrt{2E}}
\]
forces \( \cos \varphi(t) \geq \frac{1}{p} \left( \frac{p_\varphi}{\sqrt{2E}} - R \right) > -1 \). Let \( \cos \varphi_0 = \frac{1}{p} \left( \frac{p_\varphi}{\sqrt{2E}} - R \right) \) with \( 0 < \varphi_0 < \pi \). The geodesic oscillates around the outside equator of the torus with \( \varphi \) oscillating between \( \pm \varphi_0 \) while \( \dot{\theta} \) remains of fixed sign and bounded away from zero.

\[ \frac{1}{2} \rho^2 \dot{\varphi}(0)^2 < \frac{1}{2} (R - \rho)^{-2} p_\varphi^2 \]

\( \varphi_0 \) is chosen so that \( \frac{1}{2} \left( R + \rho \cos \varphi(0) \right)^{-2} p_\varphi^2 < \frac{1}{2} (R - \rho)^{-2} p_\varphi^2 \) and then choosing \( \varphi(0) \) to satisfy

\( E = \frac{1}{2} \rho^2 \dot{\varphi}(0)^2 + \frac{1}{2} \left( R + \rho \cos \varphi(0) \right)^{-2} p_\varphi^2 \). If, for example, \( \varphi(0) > 0 \), then \( \varphi(t) \) increases towards \( \pi \), but \( \dot{\varphi}(t) \) decreases towards 0 at the same time in such a way that \( \varphi(t) \) never actually achieves the value \( \pi \) (just as happened in Problem Set 2, \#3). At the same time \( \dot{\theta} \) remains of fixed sign and bounded away from zero. So the geodesic approaches the inner equator asymptotically.

\[ \frac{1}{2} \rho^2 \dot{\varphi}(t)^2 = E - \frac{1}{2} (R + \rho \cos \varphi(t))^{-2} p_\varphi^2 \geq E - \frac{1}{2} (R - \rho)^{-2} p_\varphi^2 > 0 \]
As \( \dot{\varphi}(t) \) is continuous, it remains bounded away from zero and of constant sign. Since

\[
\dot{\theta}(t) = \frac{p_{\theta}}{(R + \rho \cos \varphi(t))^2},
\]

\( \dot{\theta}(t) \) also remains bounded away from zero and of constant sign. So the geodesic wraps around the torus. The figure below shows part of such a geodesic. The heavy solid line is the portion with \( r(t) \geq R \) and the heavy dashed line is the portion with \( r(t) \leq R \).