

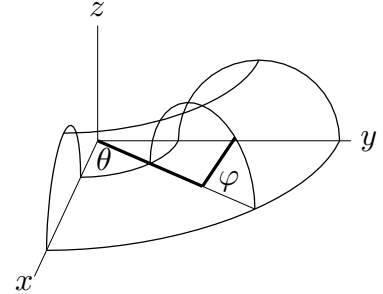
# Torus Geodesics

Let  $0 < \rho < R$  be constants. The surface in  $\mathbb{R}^3$  whose equation in cylindrical coordinates is

$$(r - R)^2 + z^2 = \rho^2$$

is a torus, which we shall call  $M$ . Use as coordinates on  $M$  two angles  $\theta$  and  $\varphi$  determined by

$$x = (R + \rho \cos \varphi) \cos \theta \quad y = (R + \rho \cos \varphi) \sin \theta \quad z = \rho \sin \varphi$$



By way of a check, observe that  $r = R + \rho \cos \varphi$  and  $z = \rho \sin \varphi$  obey the equation  $(r - R)^2 + z^2 = \rho^2$ . For any curve on  $M$

$$\begin{aligned} \dot{x}(t) &= -\dot{\varphi}(t) \rho \sin \varphi(t) \cos \theta(t) - \dot{\theta}(t) (R + \rho \cos \varphi(t)) \sin \theta(t) \\ \dot{y}(t) &= -\dot{\varphi}(t) \rho \sin \varphi(t) \sin \theta(t) + \dot{\theta}(t) (R + \rho \cos \varphi(t)) \cos \theta(t) \\ \dot{z}(t) &= \dot{\varphi}(t) \rho \cos \varphi(t) \end{aligned}$$

Thus

$$\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2 = \rho^2 \dot{\varphi}(t)^2 + (R + \rho \cos \varphi(t))^2 \dot{\theta}(t)^2$$

and the Lagrangian for free motion on  $M$  is  $L(\theta, \varphi, v_\theta, v_\varphi) = \frac{1}{2} \rho^2 v_\varphi^2 + \frac{1}{2} (R + \rho \cos \varphi)^2 v_\theta^2$ . The  $\theta$  Euler–Lagrange equation is

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial v_\theta}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \right) &= \frac{\partial L}{\partial \theta}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \\ \implies \frac{d}{dt} \left( (R + \rho \cos \varphi(t))^2 \dot{\theta}(t) \right) &= 0 \\ \implies (R + \rho \cos \varphi(t))^2 \dot{\theta}(t) &= p_\theta, \text{ constant} \end{aligned} \tag{EL}_\theta$$

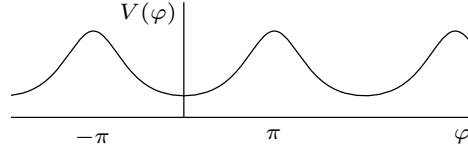
The  $\varphi$  Euler–Lagrange equation is

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial v_\varphi}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \right) &= \frac{\partial L}{\partial \varphi}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \\ \implies \rho^2 \ddot{\varphi}(t) &= -\rho (R + \rho \cos \varphi(t)) \sin \varphi(t) \dot{\theta}(t)^2 \\ \implies \rho^2 \ddot{\varphi}(t) &= -\rho (R + \rho \cos \varphi(t))^{-3} \sin \varphi(t) p_\theta^2 \end{aligned} \tag{EL}_\varphi$$

By conservation of energy

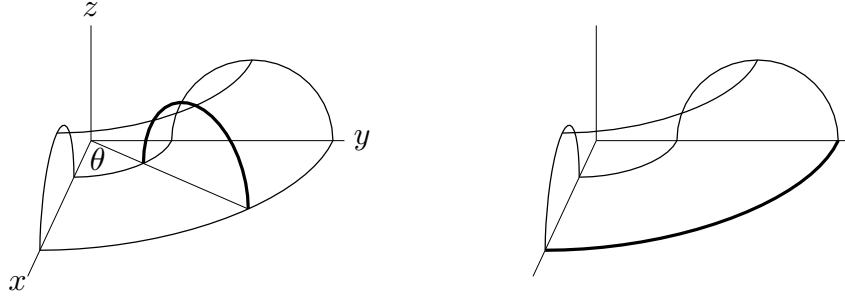
$$\begin{aligned} \frac{1}{2}\rho^2\dot{\varphi}(t)^2 + \frac{1}{2}(R + \rho \cos \varphi(t))^2\dot{\theta}(t)^2 &= E, \text{ constant} \\ \implies \frac{1}{2}\rho^2\dot{\varphi}(t)^2 &= E - \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \end{aligned} \quad (\text{E})$$

Observe that, by making appropriate choices of initial conditions, we may achieve any value of  $p_\theta \in \mathbb{R}$ . (For example, we could choose  $\theta(0) = \varphi(0) = 0$ ,  $\dot{\varphi}(0) = 0$  and  $\dot{\theta}(0) = \frac{p_\theta}{(R+\rho)^2}$ .) For any fixed  $p_\theta$ ,  $E = \frac{1}{2}\rho^2\dot{\varphi}(t)^2 + \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \geq \frac{1}{2}(R + \rho)^{-2}p_\theta^2$  since  $\cos \varphi(t) \leq 1$ . By making appropriate choices of initial conditions, we may achieve any value of  $E \geq \frac{1}{2}(R + \rho)^{-2}p_\theta^2$ . Observe that  $\frac{1}{2}\rho^2\dot{\varphi}(t)^2 + \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2$  is exactly the sum of the kinetic and potential energies for a particle of mass  $\rho^2$  moving in one dimension with potential energy  $V(\varphi) = \frac{1}{2}(R + \rho \cos \varphi)^{-2}p_\theta^2$ .



So you can develop some intuition about the behaviour of  $\varphi(t)$  by imagining what happens to a particle moving on the surface in the figure above.

- If  $p_\theta = 0$ , then, from  $(EL_\theta)$  and  $(EL_\varphi)$ ,  $\dot{\theta}(t) = \ddot{\varphi}(t) = 0$  for all  $t$  and the constant speed geodesic sweeps out a circle with  $\theta$  and  $\dot{\varphi}$  constant. In the figure on the left below, the heavy line is the top half of the geodesic.



- If  $p_\theta \neq 0$  and  $E = \frac{1}{2}(R + \rho)^{-2}p_\theta^2$ , then the condition

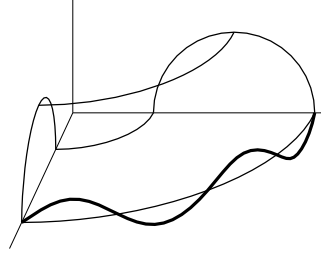
$$\frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \leq E = \frac{1}{2}(R + \rho)^{-2}p_\theta^2 \iff (R + \rho \cos \varphi(t))^{-2} \leq (R + \rho)^{-2}$$

forces  $\cos \varphi(t) \geq 1$  and hence  $\varphi(t) = 0$  for all  $t$ . The geodesic sweeps out the outside equator of the torus,  $r = R + \rho$ ,  $z = 0$ , with  $\dot{\theta}$  constant. In the figure on the right above, the heavy line is the one quarter of the geodesic.

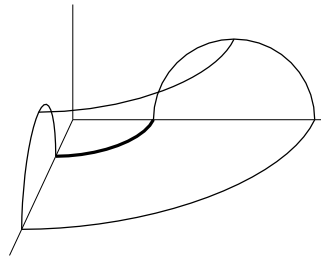
- If  $p_\theta \neq 0$  and  $\frac{1}{2}(R + \rho)^{-2}p_\theta^2 < E < \frac{1}{2}(R - \rho)^{-2}p_\theta^2$ , then the condition

$$\frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \leq E \iff (R + \rho \cos \varphi(t))^{-2} \leq \frac{2E}{p_\theta^2} \iff R + \rho \cos \varphi(t) \geq \frac{p_\theta}{\sqrt{2E}}$$

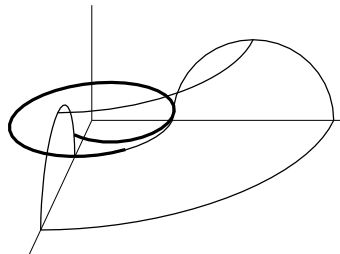
forces  $\cos \varphi(t) \geq \frac{1}{\rho} \left( \frac{p_\theta}{\sqrt{2E}} - R \right) > -1$ . Let  $\cos \varphi_0 = \frac{1}{\rho} \left( \frac{p_\theta}{\sqrt{2E}} - R \right)$  with  $0 < \varphi_0 < \pi$ . The geodesic oscillates around the outside equator of the torus with  $\varphi$  oscillating between  $\pm\varphi_0$  while  $\dot{\theta}$  remains of fixed sign and bounded away from zero.



- If  $p_\theta \neq 0$  and  $E = \frac{1}{2}(R - \rho)^{-2}p_\theta^2$ , then  $\theta(t) = \frac{p_\theta}{(R-\rho)^2}t$ ,  $\varphi(t) = \pi$  satisfies both  $(EL_\theta)$  and  $(EL_\varphi)$  and has the desired values of  $p_\theta$  and  $E$ . This geodesic sweeps out the inside equator of the torus,  $r = R - \rho$ ,  $z = 0$ .



But we may also achieve the same values of  $p_\theta$  and  $E$  by choosing some  $-\pi < \varphi(0) < \pi$  (so that  $\frac{1}{2}(R + \rho \cos \varphi(0))^{-2}p_\theta^2 < \frac{1}{2}(R - \rho)^{-2}p_\theta^2$ ) and then choosing  $\dot{\varphi}(0)$  to satisfy  $E = \frac{1}{2}\rho^2\dot{\varphi}(0)^2 + \frac{1}{2}(R + \rho \cos \varphi(0))^{-2}p_\theta^2$ . If, for example,  $\dot{\varphi}(0) > 0$ , then  $\varphi(t)$  increases towards  $\pi$ , but  $\dot{\varphi}(t)$  decreases towards 0 at the same time in such a way that  $\varphi(t)$  never actually achieves the value  $\pi$  (just as happened in Problem Set 2, #3). At the same time  $\dot{\theta}$  remains of fixed sign and bounded away from zero. So the geodesic approaches the inner equator asymptotically.



- If  $p_\theta \neq 0$  and  $E > \frac{1}{2}(R - \rho)^{-2}p_\theta^2$ , then

$$\frac{1}{2}\rho^2\dot{\varphi}(t)^2 = E - \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \geq E - \frac{1}{2}(R - \rho)^{-2}p_\theta^2 > 0$$

As  $\dot{\varphi}(t)$  is continuous, it remains bounded away from zero and of constant sign. Since

$$\dot{\theta}(t) = \frac{p_{\theta}}{(R + \rho \cos \varphi(t))^2}$$

$\dot{\theta}(t)$  also remains bounded away from zero and of constant sign. So the geodesic wraps around the torus. The figure below shows part of such a geodesic. The heavy solid line is the portion with  $r(t) \geq R$  and the heavy dashed line is the portion with  $r(t) \leq R$ .

