

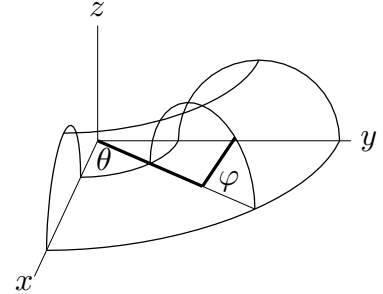
Torus Geodesics

Let $0 < \rho < R$ be constants. The surface in \mathbb{R}^3 whose equation in cylindrical coordinates is

$$(r - R)^2 + z^2 = \rho^2$$

is a torus, which we shall call M . Use as coordinates on M two angles θ and φ determined by

$$x = (R + \rho \cos \varphi) \cos \theta \quad y = (R + \rho \cos \varphi) \sin \theta \quad z = \rho \sin \varphi$$



By way of a check, observe that $r = R + \rho \cos \varphi$ and $z = \rho \sin \varphi$ obey the equation $(r - R)^2 + z^2 = \rho^2$. For any curve on M

$$\begin{aligned} \dot{x}(t) &= -\dot{\varphi}(t) \rho \sin \varphi(t) \cos \theta(t) - \dot{\theta}(t) (R + \rho \cos \varphi(t)) \sin \theta(t) \\ \dot{y}(t) &= -\dot{\varphi}(t) \rho \sin \varphi(t) \sin \theta(t) + \dot{\theta}(t) (R + \rho \cos \varphi(t)) \cos \theta(t) \\ \dot{z}(t) &= \dot{\varphi}(t) \rho \cos \varphi(t) \end{aligned}$$

Thus

$$\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2 = \rho^2 \dot{\varphi}(t)^2 + (R + \rho \cos \varphi(t))^2 \dot{\theta}(t)^2$$

and the Lagrangian for free motion on M is $L(\theta, \varphi, v_\theta, v_\varphi) = \frac{1}{2} \rho^2 v_\varphi^2 + \frac{1}{2} (R + \rho \cos \varphi)^2 v_\theta^2$. The θ Euler–Lagrange equation is

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial v_\theta}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \right) &= \frac{\partial L}{\partial \theta}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \\ \implies \frac{d}{dt} \left((R + \rho \cos \varphi(t))^2 \dot{\theta}(t) \right) &= 0 \\ \implies (R + \rho \cos \varphi(t))^2 \dot{\theta}(t) &= p_\theta, \text{ constant} \end{aligned} \tag{EL}_\theta$$

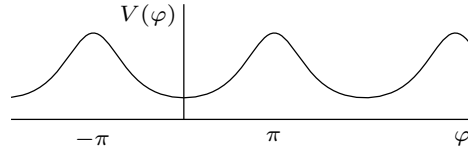
The φ Euler–Lagrange equation is

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial v_\varphi}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \right) &= \frac{\partial L}{\partial \varphi}(\theta(t), \varphi(t), \dot{\theta}(t), \dot{\varphi}(t)) \\ \implies \rho^2 \ddot{\varphi}(t) &= -\rho (R + \rho \cos \varphi(t)) \sin \varphi(t) \dot{\theta}(t)^2 \\ \implies \rho^2 \ddot{\varphi}(t) &= -\rho (R + \rho \cos \varphi(t))^{-3} \sin \varphi(t) p_\theta^2 \end{aligned} \tag{EL}_\varphi$$

By conservation of energy

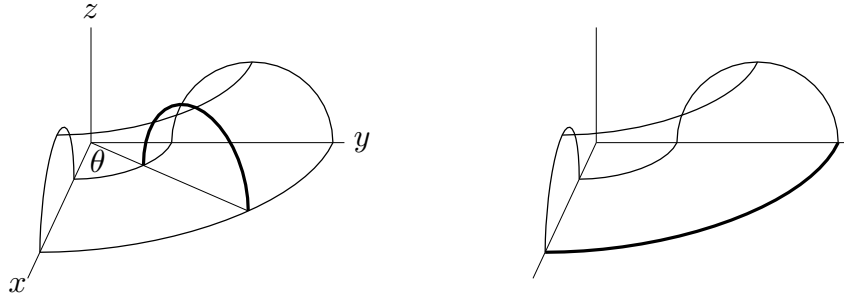
$$\begin{aligned} \frac{1}{2}\rho^2\dot{\varphi}(t)^2 + \frac{1}{2}(R + \rho \cos \varphi(t))^2\dot{\theta}(t)^2 &= E, \text{ constant} \\ \implies \frac{1}{2}\rho^2\dot{\varphi}(t)^2 &= E - \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \end{aligned} \quad (\text{E})$$

Observe that, by making appropriate choices of initial conditions, we may achieve any value of $p_\theta \in \mathbb{R}$. (For example, we could choose $\theta(0) = \varphi(0) = 0$, $\dot{\varphi}(0) = 0$ and $\dot{\theta}(0) = \frac{p_\theta}{(R+\rho)^2}$.) For any fixed p_θ , $E = \frac{1}{2}\rho^2\dot{\varphi}(t)^2 + \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \geq \frac{1}{2}(R + \rho)^{-2}p_\theta^2$ since $\cos \varphi(t) \leq 1$. By making appropriate choices of initial conditions, we may achieve any value of $E \geq \frac{1}{2}(R + \rho)^{-2}p_\theta^2$. Observe that $\frac{1}{2}\rho^2\dot{\varphi}(t)^2 + \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2$ is exactly the sum of the kinetic and potential energies for a particle of mass ρ^2 moving in one dimension with potential energy $V(\varphi) = \frac{1}{2}(R + \rho \cos \varphi)^{-2}p_\theta^2$.



So you can develop some intuition about the behaviour of $\varphi(t)$ by imagining what happens to a particle moving on the surface in the figure above.

- If $p_\theta = 0$, then, from (EL_θ) and (EL_φ) , $\dot{\theta}(t) = \ddot{\varphi}(t) = 0$ for all t and the constant speed geodesic sweeps out a circle with θ and $\dot{\varphi}$ constant. In the figure on the left below, the heavy line is the top half of the geodesic.



- If $p_\theta \neq 0$ and $E = \frac{1}{2}(R + \rho)^{-2}p_\theta^2$, then the condition

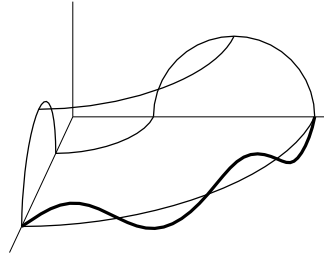
$$\frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \leq E = \frac{1}{2}(R + \rho)^{-2}p_\theta^2 \iff (R + \rho \cos \varphi(t))^{-2} \leq (R + \rho)^{-2}$$

forces $\cos \varphi(t) \geq 1$ and hence $\varphi(t) = 0$ for all t . The geodesic sweeps out the outside equator of the torus, $r = R + \rho$, $z = 0$, with $\dot{\theta}$ constant. In the figure on the right above, the heavy line is the one quarter of the geodesic.

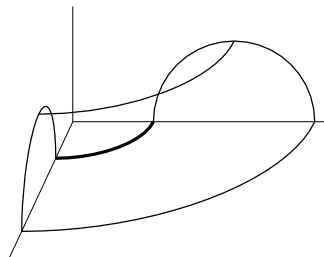
- If $p_\theta \neq 0$ and $\frac{1}{2}(R + \rho)^{-2}p_\theta^2 < E < \frac{1}{2}(R - \rho)^{-2}p_\theta^2$, then the condition

$$\frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \leq E \iff (R + \rho \cos \varphi(t))^{-2} \leq \frac{2E}{p_\theta^2} \iff R + \rho \cos \varphi(t) \geq \frac{p_\theta}{\sqrt{2E}}$$

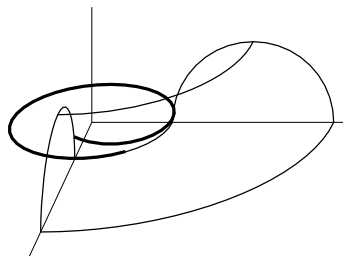
forces $\cos \varphi(t) \geq \frac{1}{\rho} \left(\frac{p_\theta}{\sqrt{2E}} - R \right) > -1$. Let $\cos \varphi_0 = \frac{1}{\rho} \left(\frac{p_\theta}{\sqrt{2E}} - R \right)$ with $0 < \varphi_0 < \pi$. The geodesic oscillates around the outside equator of the torus with φ oscillating between $\pm\varphi_0$ while $\dot{\theta}$ remains of fixed sign and bounded away from zero.



- If $p_\theta \neq 0$ and $E = \frac{1}{2}(R - \rho)^{-2}p_\theta^2$, then $\theta(t) = \frac{p_\theta}{(R-\rho)^2}t$, $\varphi(t) = \pi$ satisfies both (EL_θ) and (EL_φ) and has the desired values of p_θ and E . This geodesic sweeps out the inside equator of the torus, $r = R - \rho$, $z = 0$.



But we may also achieve the same values of p_θ and E by choosing some $-\pi < \varphi(0) < \pi$ (so that $\frac{1}{2}(R + \rho \cos \varphi(0))^{-2}p_\theta^2 < \frac{1}{2}(R - \rho)^{-2}p_\theta^2$) and then choosing $\dot{\varphi}(0)$ to satisfy $E = \frac{1}{2}\rho^2\dot{\varphi}(0)^2 + \frac{1}{2}(R + \rho \cos \varphi(0))^{-2}p_\theta^2$. If, for example, $\dot{\varphi}(0) > 0$, then $\varphi(t)$ increases towards π , but $\dot{\varphi}(t)$ decreases towards 0 at the same time in such a way that $\varphi(t)$ never actually achieves the value π (just as happened in Problem Set 2, #3). At the same time $\dot{\theta}$ remains of fixed sign and bounded away from zero. So the geodesic approaches the inner equator asymptotically.



- If $p_\theta \neq 0$ and $E > \frac{1}{2}(R - \rho)^{-2}p_\theta^2$, then

$$\frac{1}{2}\rho^2\dot{\varphi}(t)^2 = E - \frac{1}{2}(R + \rho \cos \varphi(t))^{-2}p_\theta^2 \geq E - \frac{1}{2}(R - \rho)^{-2}p_\theta^2 > 0$$

As $\dot{\varphi}(t)$ is continuous, it remains bounded away from zero and of constant sign. Since

$$\dot{\theta}(t) = \frac{p_{\theta}}{(R + \rho \cos \varphi(t))^2}$$

$\dot{\theta}(t)$ also remains bounded away from zero and of constant sign. So the geodesic wraps around the torus. The figure below shows part of such a geodesic. The heavy solid line is the portion with $r(t) \geq R$ and the heavy dashed line is the portion with $r(t) \leq R$.

