A Summary of Rigid Body Formulae

By definition, a rigid body is a family of N particles, with the mass of particle number i denoted m_i and with the position of particle number i at time t denoted $\vec{x}^{(i)}(t)$, together with sufficiently many constraints of the form $\left|\vec{x}^{(i)}(t)-\vec{x}^{(j)}(t)\right| = \ell_{i,j}$, constant, that the positions of all particles at time t are uniquely determined by

(i) the position of one point $\vec{c}(t)$ fixed with respect to the body and

(ii) three mutually perpendicular unit vectors $\hat{\imath}_1(t)$, $\hat{\imath}_2(t)$, $\hat{\imath}_3(t)$, forming a right handed triple, that are also fixed with respect to the body.

That is, for each $1 \le i \le N$, particle number i has three coordinates $\vec{X}^{(i)} = \big(X_1^{(i)}, X_2^{(i)}, X_3^{(i)}\big)$, which are fixed for all time, such that

$$\begin{aligned}\vec{x}^{(i)}(t) &= \vec{c}(t) + X_1^{(i)}\hat{\imath}_1(t) + X_2^{(i)}\hat{\imath}_2(t) + X_3^{(i)}\hat{\imath}_3(t) = \vec{c}(t) + R(t)\vec{X}^{(i)} \\ &\text{where } R(t) = \big[\hat{\imath}_1(t)\ \hat{\imath}_2(t)\ \hat{\imath}_3(t)\big] \in SO(3)\end{aligned} \qquad (\textbf{position})$$

One refers to $\vec{x}^{(i)}$ as the position of particle number i expressed in laboratory coordinates, and $\vec{X}^{(i)}$ as the position of particle number i expressed in body coordinates. One often chooses $\vec{c}(t)$ to be the centre of mass. That is,

$$\vec{c}(t) = \tfrac{1}{\mu}\sum_{i=1}^{N} m_i\, \vec{x}^{(i)}(t) \qquad \text{where } \mu = \text{total mass} = \sum_{i=1}^{N} m_i$$

It is also quite common to impose the additional constraint that $\vec{c}(t) = 0$ for all time. The velocity of particle number i is

$$\begin{aligned}\dot{\vec{x}}^{(i)}(t) &= \dot{\vec{c}}(t) + \dot{R}(t)\vec{X}^{(i)} \\ &= \dot{\vec{c}}(t) + \dot{R}(t)R(t)^{-1}\big(\vec{x}^{(i)}(t) - \vec{c}(t)\big) \\ &= \dot{\vec{c}}(t) + \vec{\omega}(t) \times \big(\vec{x}^{(i)}(t) - \vec{c}(t)\big) \\ &= \dot{\vec{c}}(t) + \big(R(t)\vec{\Omega}(t)\big) \times \big(\vec{x}^{(i)}(t) - \vec{c}(t)\big) \\ &= \dot{\vec{c}}(t) + R(t)\,\big(\vec{\Omega}(t) \times \vec{X}^{(i)}\big)\end{aligned} \qquad (\textbf{velocity})$$

where $\vec{\omega}(t)$ is the angular velocity of the body expressed in laboratory coordinates, and $\Omega(t)$ is the angular velocity of the body expressed in body coordinates. Precisely,

$$\begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix} = \dot{R}(t)R(t)^{-1} \qquad \vec{\omega}(t) = \Omega_1(t)\hat{\imath}_1(t) + \Omega_2(t)\hat{\imath}_2(t) + \Omega_3(t)\hat{\imath}_3(t) = R(t)\vec{\Omega}(t)$$

(**angular velocity**)

Assuming that $\vec{c}(t)$ is the centre of mass of the body, the kinetic energy of the body is

$$KE = \tfrac{1}{2}\mu\dot{\vec{c}}(t)^2 + \tfrac{1}{2}\vec{\Omega}(t)\cdot\mathcal{I}\vec{\Omega}(t) \qquad (\textbf{kinetic energy})$$

where the matrix $\mathcal{I}$, which is time independent and called the inertia tensor, is determined by

$$\vec{u}\cdot\mathcal{I}\vec{v} = \sum_{i=1}^{N} m_i\big[\vec{u}\times\vec{X}^{(i)}\big]\cdot\big[\vec{v}\times\vec{X}^{(i)}\big]$$

In particular, for each $1 \le j,k \le 3$, the j,k matrix element of $\mathcal{I}$ is

$$\mathcal{I}_{j,k} = \sum_{i=1}^{N} m_i \big[\vec{X}^{(i)} \cdot \vec{X}^{(i)}\, \delta_{j,k} - X_j^{(i)} X_k^{(i)}\big]$$

If $\vec{c}(t) \equiv 0$, the angular momentum

$$\begin{aligned} \vec{m} &= \sum_{i=1}^{N} m_i\, \vec{x}^{(i)}(t) \times \dot{\vec{x}}^{(i)}(t) \\ &= R(t)\, \mathcal{I}\, \vec{\Omega}(t) \\ &= R(t)\, \vec{M}(t) \end{aligned} \qquad \textbf{(angular momentum)}$$

where $\vec{M}(t) = \mathcal{I}\vec{\Omega}(t)$ is the angular momentum in body coordinates. For free motion, angular momentum is conserved so that

$$0 = \tfrac{d}{dt}\vec{m} = R(t)\dot{\vec{M}}(t) + \dot{R}(t)\vec{M}(t) = R(t)\dot{\vec{M}}(t) + \vec{\omega}(t) \times (R(t)\vec{M}(t)) = R(t)\big[\dot{\vec{M}}(t) + \vec{\Omega}(t) \times \vec{M}(t)\big]$$

Thus

$$\dot{\vec{M}}(t) = \vec{M}(t) \times \vec{\Omega}(t) = \vec{M}(t) \times \mathcal{I}^{-1}\vec{M}(t) \qquad \textbf{(Euler's equations)}$$

The matrix $\mathcal{I}$ is symmetric. So we may choose a coordinate system in which it is diagonal, with its eigenvalues, I_1, I_2, I_3, on the diagonal. Then

$$\dot{M}_1(t) = \tfrac{I_2 - I_3}{I_2 I_3} M_2(t) M_3(t) \qquad \dot{M}_2(t) = \tfrac{I_3 - I_1}{I_1 I_3} M_1(t) M_3(t) \qquad \dot{M}_3(t) = \tfrac{I_1 - I_2}{I_1 I_2} M_1(t) M_2(t)$$