A Summary of Rigid Body Formulae

By definition, a rigid body is a family of $N$ particles, with the mass of particle number $i$ denoted $m_i$ and with the position of particle number $i$ at time $t$ denoted $\vec{x}^{(i)}(t)$, together with sufficiently many constraints of the form $|\vec{x}^{(i)}(t) - \vec{x}^{(j)}(t)| = \ell_{i,j}$, constant, that the positions of all particles at time $t$ are uniquely determined by

(i) the position of one point $\vec{c}(t)$ fixed with respect to the body and
(ii) three mutually perpendicular unit vectors $\hat{i}_1(t), \hat{i}_2(t), \hat{i}_3(t)$, forming a right handed triple, that are also fixed with respect to the body.

That is, for each $1 \leq i \leq N$, particle number $i$ has three coordinates $\vec{X}^{(i)} = (X_1^{(i)}, X_2^{(i)}, X_3^{(i)})$, which are fixed for all time, such that

$$\vec{x}^{(i)}(t) = \vec{c}(t) + X_1^{(i)}\hat{i}_1(t) + X_2^{(i)}\hat{i}_2(t) + X_3^{(i)}\hat{i}_3(t) = \vec{c}(t) + R(t)\vec{X}^{(i)}$$

where $R(t) = [\hat{i}_1(t) \hat{i}_2(t) \hat{i}_3(t)] \in SO(3)$

One refers to $\vec{x}^{(i)}$ as the position of particle number $i$ expressed in laboratory coordinates, and $\vec{X}^{(i)}$ as the position of particle number $i$ expressed in body coordinates. One often chooses $\vec{c}(t)$ to be the centre of mass. That is,

$$\vec{c}(t) = \frac{1}{\mu} \sum_{i=1}^{N} m_i \vec{x}^{(i)}(t) \quad \text{where } \mu = \text{total mass} = \sum_{i=1}^{N} m_i$$

It is also quite common to impose the additional constraint that $\vec{c}(t) = 0$ for all time. The velocity of particle number $i$ is

$$\dot{x}^{(i)}(t) = \ddot{\vec{c}}(t) + \dot{R}(t)\vec{X}^{(i)}$$

$$= \ddot{\vec{c}}(t) + \dot{R}(t)R(t)^{-1}(\vec{x}^{(i)}(t) - \vec{c}(t))$$

$$= \ddot{\vec{c}}(t) + \vec{\omega}(t) \times (\vec{x}^{(i)}(t) - \vec{c}(t))$$

$$= \ddot{\vec{c}}(t) + (R(t)\vec{\Omega}(t)) \times (\vec{x}^{(i)}(t) - \vec{c}(t))$$

$$= \ddot{\vec{c}}(t) + R(t)(\vec{\Omega}(t) \times \vec{X}^{(i)})$$

(velocity)

where $\vec{\omega}(t)$ is the angular velocity of the body expressed in laboratory coordinates, and $\vec{\Omega}(t)$ is the angular velocity of the body expressed in body coordinates. Precisely,

$$\begin{bmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{bmatrix} = R(t)R(t)^{-1} \quad \vec{\omega}(t) = \vec{\Omega}(t)\hat{i}_1(t) + \vec{\Omega}(t)\hat{i}_2(t) + \vec{\Omega}(t)\hat{i}_3(t) = R(t)\vec{\Omega}(t)$$

(angular velocity)

Assuming that $\vec{c}(t)$ is the centre of mass of the body, the kinetic energy of the body is

$$KE = \frac{1}{2} \mu \dot{\vec{c}}(t)^2 + \frac{1}{2} \vec{\Omega}(t) \cdot \vec{\Omega}(t)$$

(kinetic energy)

where the matrix $\mathcal{I}$, which is time independent and called the inertia tensor, is determined by

$$\vec{u} \cdot \mathcal{I} \vec{v} = \sum_{i=1}^{N} m_i [\vec{u} \times \vec{X}^{(i)}] : [\vec{v} \times \vec{X}^{(i)}]$$
In particular, for each \(1 \leq j, k \leq 3\), the \(j, k\) matrix element of \(\mathcal{I}\) is

\[
\mathcal{I}_{j,k} = \sum_{i=1}^{N} m_i \left[ \vec{X}^{(i)} \cdot \vec{X}^{(i)} \delta_{j,k} - X_j^{(i)} X_k^{(i)} \right]
\]

If \(\vec{c}(t) \equiv 0\), the angular momentum

\[
\vec{m} = \sum_{i=1}^{N} m_i \vec{x}^{(i)}(t) \times \vec{\omega}^{(i)}(t)
\]

\[
= R(t) \mathcal{I} \vec{\Omega}(t)
\]

\[
= R(t) \vec{\Omega}(t) \vec{M}(t)
\]

(angular momentum)

where \(\vec{M}(t) = \mathcal{I} \vec{\Omega}(t)\) is the angular momentum in body coordinates. For free motion, angular momentum is conserved so that

\[
0 = \frac{d}{dt} \vec{m} = R(t) \dot{\vec{M}}(t) + \dot{R}(t) \vec{M}(t) = R(t) \dot{\vec{M}}(t) + \vec{\omega}(t) \times (R(t) \vec{M}(t)) = R(t) \left[ \dot{\vec{M}}(t) + \vec{\Omega}(t) \times \vec{M}(t) \right]
\]

Thus

\[
\dot{\vec{M}}(t) = \vec{M}(t) \times \vec{\Omega}(t) = \vec{M}(t) \times \mathcal{I}^{-1} \dot{\vec{M}}(t)
\]

(Euler’s equations)

The matrix \(\mathcal{I}\) is symmetric. So we may choose a coordinate system in which it is diagonal, with its eigenvalues, \(I_1, I_2, I_3\), on the diagonal. Then

\[
\dot{\vec{M}}(t) = \begin{pmatrix} I_1-I_2 & I_2 & 0 \\ I_2 & I_3-I_1 & 0 \\ 0 & 0 & I_1-I_2 \end{pmatrix} \vec{M}(t)
\]

\[
\dot{\vec{M}}(t) = \begin{pmatrix} \frac{I_1-I_2}{I_2} & \frac{I_2}{I_2} & 0 \\ \frac{I_2}{I_2} & \frac{I_3-I_1}{I_3} & 0 \\ 0 & 0 & \frac{I_1-I_2}{I_3} \end{pmatrix} \vec{M}(t)
\]