The Principle of Least Action

We have seen in class that Newton's law, $\frac{1}{2}m\ddot{\vec{x}}(t) = -\vec{\nabla}U\big(\vec{x}(t)\big)$ for the motion of a particle in a potential well is equivalent to the stationarity of the action

$$S_{[t_1,t_2]}\big\{\vec{x}(t)\big\} = \int_{t_1}^{t_2} L\big(\vec{x}(t), \dot{\vec{x}}(t)\big)\ dt \quad \text{with Lagrangian} \quad L(\vec{x},\vec{v}) = \tfrac{1}{2}m\vec{v}^{\,2} - U(\vec{x})$$

This is known as the "Principle of Least Action" (or Hamilton's Principle). Here is the Theorem which provides the motivation for that name.

Theorem. *Let $\vec{x}(t)$ be a stationary curve for the Lagrangian $L(\vec{x},\vec{v}) = \frac{1}{2}m\vec{v}^{\,2} - U(\vec{x})$ on the interval $a \le t \le b$. Then there exists a $\delta > 0$ (depending on m, the potential $U(\vec{x})$ and the curve $\vec{x}(t)$) such that whenever*

- ○ *$a \le t_1 < t_2 \le b$ with $t_2 - t_1 \le \delta$*
- ○ *$\vec{\zeta} : [t_1, t_2] \to \mathbb{R}^n$ is a C^∞ curve obeying $\vec{\zeta}(t_1) = \vec{0}$, $\vec{\zeta}(t_2) = \vec{0}$ and $|\vec{\zeta}(t)| \le 1$ for all $t_1 \le t \le t_2$*

then

$$S_{[t_1,t_2]}\big\{\vec{x}(t) + \vec{\zeta}(t)\big\} \ge S_{[t_1,t_2]}\big\{\vec{x}(t)\big\}$$

Thus $\vec{x}(t)$ is a local minimum of the action for short time intervals.

Proof: We wish to show that

$$\begin{aligned}
\Delta &\equiv S_{[t_1,t_2]}\big\{\vec{x}(t) + \vec{\zeta}(t)\big\} - S_{[t_1,t_2]}\big\{\vec{x}(t)\big\} \\
&= \int_{t_1}^{t_2} dt\ \Big[\tfrac{1}{2}m\dot{\vec{\zeta}}(t)^2 + m\,\dot{\vec{x}}(t)\cdot\dot{\vec{\zeta}}(t) - U\big(\vec{x}(t) + \vec{\zeta}(t)\big) + U\big(\vec{x}(t)\big)\Big]
\end{aligned}$$

is positive. Taylor expanding $f(\eta) = -U\big(\vec{x}(t)+\eta\vec{\zeta}(t)\big)+U\big(\vec{x}(t)\big)$ to second order in η, including the remainder, and then setting $\eta = 1$ gives

$$\begin{aligned}
-U\big(\vec{x} + \vec{\zeta}\,\big) + U\big(\vec{x}\,\big) &= f(1) = f(0) + f'(0) + \tfrac{1}{2}f''(\bar{\eta}) \\
&= -\vec{\zeta}\cdot\tfrac{\partial U}{\partial \vec{x}}(\vec{x}) - \tfrac{1}{2}\sum_{i,j=1}^{n} \zeta_i\zeta_j \tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial x_j} U(\vec{x} + \bar{\eta}\vec{\zeta}\,)
\end{aligned}$$

for some $0 \le \bar{\eta} \le 1$ (depending on t). For all $0 \le \bar{\eta} \le 1$ and all $t_1 \le t \le t_2$ and all $\vec{\zeta}$ of magnitude at most one, $\vec{x}(t) + \bar{\eta}\vec{\zeta}$ lies within distance one of the compact set $\mathcal{X} = \big\{\ \vec{x}(t)\ \big|\ t_1 \le t \le t_2\ \big\}$. The set of all points of distance at most one from $\mathcal{X}$ is again compact. Hence there is a finite constant M such that

$$-U\big(\vec{x} + \vec{\zeta}\big) + U\big(\vec{x}\big) \ge -\vec{\zeta}\cdot\tfrac{\partial U}{\partial \vec{x}}(\vec{x}) - M|\vec{\zeta}|^2$$

Hence

$$\Delta \ge \int_{t_1}^{t_2} dt\ \Big[\tfrac{1}{2}m\dot{\vec{\zeta}}(t)^2 - M\vec{\zeta}(t)^2\Big] + \int_{t_1}^{t_2} dt\ \Big[m\,\dot{\vec{x}}(t)\cdot\dot{\vec{\zeta}}(t) - \vec{\zeta}(t)\cdot\tfrac{\partial U}{\partial \vec{x}}\big(\vec{x}(t)\big)\Big] \tag{1}$$

Now the second integral is exactly $\frac{d}{d\varepsilon}S_{[t_1,t_2]}\big\{\vec{x}(t) + \varepsilon\vec{\zeta}(t)\big\}\Big|_{\varepsilon=0}$ and hence is zero by the stationarity of $\vec{x}(t)$. (Note that, since $\vec{x}(t)$ is stationary on $[a, b]$, it obeys the Euler–Lagrange equation on $[a, b]$. So it also obeys the Euler–Lagrange equation on $[t_1,t_2]\subset[a,b]$ and so is stationary for $[t_1,t_2]$ too.) For the first integral we use

$$\begin{aligned}
\big|\zeta_i(t)\big| &= \left|\int_{t_1}^{t}\dot\zeta_i(s)\ ds\right| \qquad \text{(Recall that } \vec\zeta(t_1)=0.\text{)}\\
&\le \left[\int_{t_1}^{t} 1\ ds\right]^{1/2}\left[\int_{t_1}^{t}\dot\zeta_i(s)^2\ ds\right]^{1/2} \qquad \text{(by the Cauchy–Schwarz inequality)}\\
&= \big[t-t_1\big]^{1/2}\left[\int_{t_1}^{t}\dot\zeta_i(s)^2\ ds\right]^{1/2}
\end{aligned}$$

which implies that

$$\vec\zeta(t)^2 \le \big[t-t_1\big]\int_{t_1}^{t}\dot{\vec\zeta}(s)^2\ ds \le \big[t-t_1\big]\int_{t_1}^{t_2}\dot{\vec\zeta}(s)^2\ ds$$

and

$$\int_{t_1}^{t_2}\vec\zeta(t)^2\ dt \le \int_{t_1}^{t_2}\dot{\vec\zeta}(s)^2\ ds\int_{t_1}^{t_2}\big[t-t_1\big]\ dt = \tfrac12\big[t_2-t_1\big]^2\int_{t_1}^{t_2}\dot{\vec\zeta}(s)^2\ ds \tag{2}$$

Substituting (2) into (1) gives

$$\Delta \ge \int_{t_1}^{t_2} dt\ \tfrac12\Big(m - M\big[t_2-t_1\big]^2\Big)\dot{\vec\zeta}(t)^2 \ge 0$$

provided $m - M\big[t_2-t_1\big]^2 \ge 0$, or equivalently,

$$t_2 - t_1 \le \sqrt{\tfrac{m}{M}} \equiv \delta$$

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Example. We illustrate this result using dimension $n=1$, the harmonic oscillator Lagrangian $L(x,v)=\frac12\big[v^2-x^2\big]$, interval $[a,b]=[t_1,t_2]=[0,\mu\pi]$ for noninteger positive μ and boundary conditions $x(0)=x(\mu\pi)=0$. The Euler–Lagrange equation is $\ddot x + x = 0$. The general solution to this equation is $x(t)=A\sin t + B\cos t$. The condition $x(0)=0$ forces $B=0$ and the condition $x(\mu\pi)=A\sin(\mu\pi)=0$ forces $A=0$, so the only stationary curve is $x(t)=0$ and the corresponding action $S\big\{x(t)\big\}=0$. Denote

$$\zeta_{\varepsilon,k} = \varepsilon\sin\big(\tfrac{k}{\mu}t\big)$$

Then

$$\begin{aligned}
S\big\{\zeta_{\varepsilon,k}\big\} &= \int_0^{\mu\pi}\tfrac12\Big[\varepsilon^2\big(\tfrac{k}{\mu}\big)^2\cos^2\big(\tfrac{k}{\mu}t\big) - \varepsilon^2\sin^2\big(\tfrac{k}{\mu}t\big)\Big]\ dt\\
&= \tfrac12\mu\varepsilon^2\int_0^{\pi}\Big[\big(\tfrac{k}{\mu}\big)^2\cos^2(ks) - \sin^2(ks)\Big]\ ds \qquad \text{with } t=\mu s\\
&= \tfrac{\pi}{4}\mu\varepsilon^2\Big[\big(\tfrac{k}{\mu}\big)^2 - 1\Big]
\end{aligned}$$

If $\mu>1$ (i.e. the time interval is longer than π), this is negative for $k<\mu$ and positive for $k>\mu$. So we have a saddle point. If $\mu<1$, this is positive for all $k\in\mathbb{N}$, which is consistent with having a local minimum.