

Phase Plane Analysis of the Undamped Pendulum

Phase plane analysis is a commonly used technique for determining the qualitative behaviour of solutions of systems of ODE's in low dimensions. To illustrate the technique, we consider the equation $\frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin \theta = 0$, which models the undamped, nonlinear, planar pendulum. We first convert this second order equation into a first order system by introducing the variables $x = \theta$ and $y = \frac{d\theta}{dt}$. Then

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{\ell} \sin x \end{bmatrix} \quad (*)$$

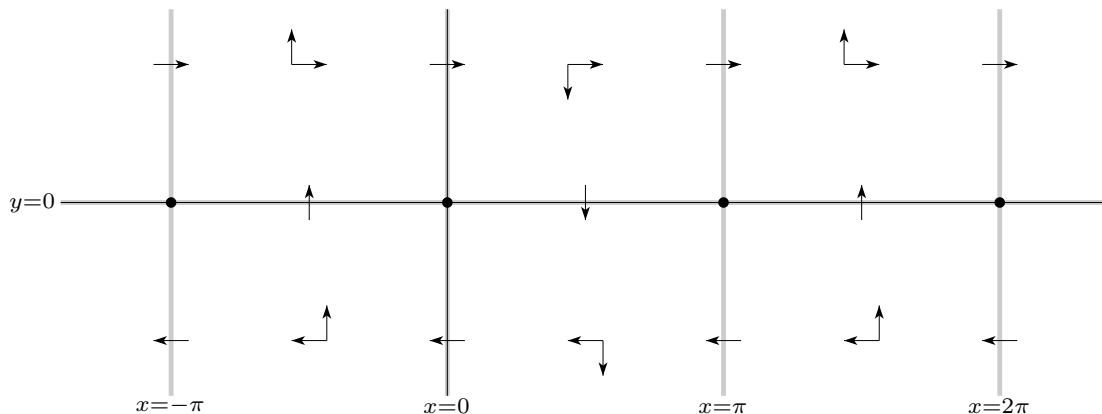
Physically, x represents the position of the pendulum and y represents its angular velocity. But I find it useful to temporarily ignore this physical significance and to pretend that (x, y) gives the position of a tiny car driving in the xy -plane. The differential equation $(*)$ then tells us that whenever the car is at the point (x, y) it has velocity $(y, -\frac{g}{\ell} \sin x)$. To start the analysis we determine at which points (x, y) in the plane we have $\frac{dx}{dt} = 0$; at which points $\frac{dx}{dt} > 0$; and at which points $\frac{dx}{dt} < 0$. And do the same for $\frac{dy}{dt}$.

$$\frac{dx}{dt} \left\{ \begin{array}{l} = 0 \quad \text{if } y = 0 \\ < 0 \quad \text{if } y < 0 \\ > 0 \quad \text{if } y > 0 \end{array} \right\} \quad \frac{dy}{dt} \left\{ \begin{array}{l} = 0 \quad \text{if } x = n\pi, n \in \mathbb{Z} \\ < 0 \quad \text{if } x \in [n\pi, (n+1)\pi], n \text{ even} \\ > 0 \quad \text{if } x \in [n\pi, (n+1)\pi], n \text{ odd} \end{array} \right\}$$

This tells us that, as far as the horizontal component of motion is concerned,

- on the x -axis, there is no horizontal component of motion – the motion is purely vertical
- in the upper half plane the horizontal component of the velocity is positive i.e. the motion is rightward
- in the lower half plane the horizontal component of the velocity is negative i.e. the motion is leftward

and we have the analogous conclusions about the vertical component of motion. Here is a sketch which encodes this data.



The dots are points at which $\frac{dx}{dt} = \frac{dy}{dt} = 0$. Trajectories which start at those points remain there for ever. They are called critical points. The critical points $(n\pi, 0)$ with n even, correspond to motions of the pendulum for which the pendulum remains in the configuration \downarrow for ever. The critical points $(n\pi, 0)$ with n odd, correspond to motions of the pendulum for which the pendulum remains in the configuration \uparrow for ever.

Other trajectories can be found (or at least guessed at) by “following the arrows”. For example, suppose that we start at $t = 0$ with our tiny car at a point (x, y) with $x > -\pi$, but very close to $-\pi$, and $y > 0$, but very close to $y = 0$. Looking at the arrows, we see that as time progresses both x and y will increase (so that the car moves upwards and to the right) until the car reaches the y -axis. It crosses the y -axis horizontally. Then y starts to decrease while x continues to increase. So the car moves downwards and to the right. This continues until the car hits either the x -axis or the line $x = \pi$. In the former case, the car crosses the x -axis vertically downwards and then heads downwards to the left. (The pendulum has reached its maximum angle and has stopped and reversed its direction of swing.) In the latter case, the car crosses the line $x = \pi$ horizontally and moves upwards and to the right. (The pendulum has swung all the way over the top and continues with the same direction of swing.) There is one special case in which the car approaches the x -axis and the line $x = \pi$ at the same time. That is, it heads directly towards $(\pi, 0)$. In this case, as the car gets closer and closer to $(\pi, 0)$, it slows down so that it never actually gets to $(\pi, 0)$. (The pendulum approaches \uparrow , but never actually gets there.) The figure below shows a number of representative trajectories.

