

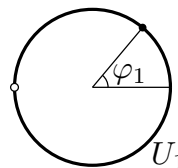
Examples of Manifolds

Example 1 (Open Subset of \mathbb{R}^n) Any open subset, \mathcal{O} , of \mathbb{R}^n is a manifold of dimension n . One possible atlas is $\mathcal{A} = \{(\mathcal{O}, \varphi_{\text{id}})\}$, where φ_{id} is the identity map. That is, $\varphi_{\text{id}}(\mathbf{x}) = \mathbf{x}$. Of course one possible choice of \mathcal{O} is \mathbb{R}^n itself.

Example 2 (The Circle) The circle $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is a manifold of dimension one. One possible atlas is $\mathfrak{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ where

$$U_1 = S^1 \setminus \{(-1, 0)\} \quad \varphi_1(x, y) = \arctan \frac{y}{x} \text{ with } -\pi < \varphi_1(x, y) < \pi$$

$$U_2 = S^1 \setminus \{(1, 0)\} \quad \varphi_2(x, y) = \arctan \frac{y}{x} \text{ with } 0 < \varphi_2(x, y) < 2\pi$$



Example 3 (S^n) The n -sphere $S^n = \{ \mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1 \}$ is a manifold of dimension n . One possible atlas is $\mathfrak{A}_1 = \{ (U_i, \varphi_i), (V_i, \psi_i) \mid 1 \leq i \leq n+1 \}$ where, for each $1 \leq i \leq n+1$,

$$U_i = \{ (x_1, \dots, x_{n+1}) \in S^n \mid x_i > 0 \} \quad \varphi_i(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$$

$$V_i = \{ (x_1, \dots, x_{n+1}) \in S^n \mid x_i < 0 \} \quad \psi_i(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$$

So both φ_i and ψ_i project onto \mathbb{R}^n , viewed as the hyperplane $x_i = 0$. Another possible atlas is

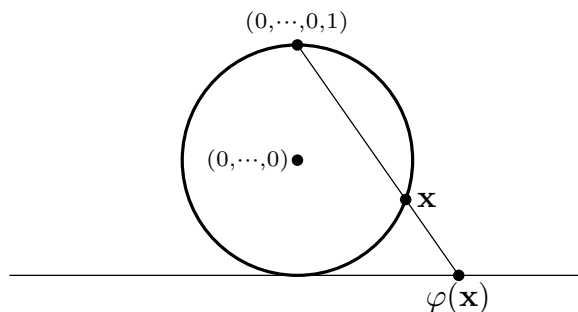
$$\mathfrak{A}_2 = \{ (S^n \setminus \{(0, \dots, 0, 1)\}, \varphi), (S^n \setminus \{(0, \dots, 0, -1)\}, \psi) \}$$

where

$$\varphi(x_1, \dots, x_{n+1}) = \left(\frac{2x_1}{1-x_{n+1}}, \dots, \frac{2x_n}{1-x_{n+1}} \right)$$

$$\psi(x_1, \dots, x_{n+1}) = \left(\frac{2x_1}{1+x_{n+1}}, \dots, \frac{2x_n}{1+x_{n+1}} \right)$$

are the stereographic projections from the north and south poles, respectively.

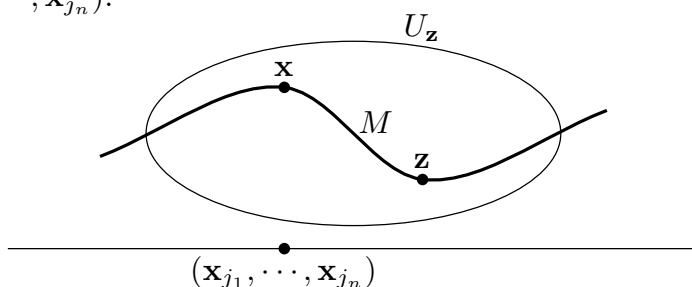


Both φ and ψ have range \mathbb{R}^n . So we can think of S^n as \mathbb{R}^n plus an additional single “point at infinity”.

Example 4 (Surfaces) Any smooth n -dimensional surface in \mathbb{R}^{n+m} is an n -dimensional manifold. When we say that M is an n -dimensional surface in \mathbb{R}^{n+m} , we mean that M is a subset of \mathbb{R}^{n+m} with the property that for each $\mathbf{z} \in M$, there are

- a neighbourhood $U_{\mathbf{z}}$ of \mathbf{z} in \mathbb{R}^{n+m}
- n integers $1 \leq j_1 < j_2 < \cdots < j_n \leq n+m$
- and m C^∞ functions $f_k(\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n})$, $k \in \{1, \dots, n+m\} \setminus \{j_1, \dots, j_n\}$

such that the point $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_{n+m}) \in U_{\mathbf{z}}$ is in M if and only if $\mathbf{x}_k = f_k(\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n})$ for all $k \in \{1, \dots, n+m\} \setminus \{j_1, \dots, j_n\}$. That is, we may choose n components \mathbf{x}_{j_ℓ} , $1 \leq \ell \leq n$ of \mathbf{x} as local coordinates on M near \mathbf{z} . On M , near \mathbf{z} , the remaining components of \mathbf{x} are C^∞ functions of \mathbf{x}_{j_ℓ} , $1 \leq \ell \leq n$. Of course, an atlas is $\mathfrak{A} = \{ (U_{\mathbf{z}}, \varphi_{\mathbf{z}}) \mid \mathbf{z} \in M \}$, with each $\varphi_{\mathbf{z}}(\mathbf{x}) = (\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n})$.



Equivalently, M is an n -dimensional surface in \mathbb{R}^{n+m} , if, for each $\mathbf{z} \in M$, there are

- a neighbourhood $U_{\mathbf{z}}$ of \mathbf{z} in \mathbb{R}^{n+m}
- and m C^∞ functions $g_k : U_{\mathbf{z}} \rightarrow \mathbb{R}$, such that the vectors $\{ \nabla g_k(\mathbf{z}) \mid 1 \leq k \leq m \}$ are linearly independent

such that the point $\mathbf{x} \in U_{\mathbf{z}}$ is in M if and only if $g_k(\mathbf{x}) = 0$ for all $1 \leq k \leq m$. To get from the g_k 's to the f_k 's one need only invoke (possibly after renumbering the components of \mathbf{x})

Implicit Function Theorem

Let $m, n \in \mathbb{N}$ and let $U \subset \mathbb{R}^{n+m}$ be an open set. Let $\mathbf{g} : U \rightarrow \mathbb{R}^m$ be C^∞ with $\mathbf{g}(\mathbf{z}) = 0$ for some $\mathbf{z} \in U$. Assume that $\det \left[\frac{\partial \mathbf{g}_i}{\partial \mathbf{x}_{n+j}}(\mathbf{z}) \right]_{1 \leq i, j \leq m} \neq 0$. Write $\mathbf{a} = (\mathbf{z}_1, \dots, \mathbf{z}_n)$ and $\mathbf{b} = (\mathbf{z}_{n+1}, \dots, \mathbf{z}_{n+m})$. Then there exist open sets $V \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^n$ with $\mathbf{a} \in W$ and $\mathbf{z} = (\mathbf{a}, \mathbf{b}) \in V$ such that

for each $\mathbf{x} \in W$, there is a unique $(\mathbf{x}, \mathbf{y}) \in V$ such that $\mathbf{g}(\mathbf{x}, \mathbf{y}) = 0$.

If the \mathbf{y} above is denoted $\mathbf{f}(\mathbf{x})$, then $\mathbf{f} : W \rightarrow \mathbb{R}^m$ is C^∞ , $\mathbf{f}(\mathbf{a}) = \mathbf{b}$ and $\mathbf{g}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = 0$ for all $\mathbf{x} \in W$.

The n -sphere S^n is the n -dimensional surface in \mathbb{R}^{n+1} given implicitly by the equation $g(\mathbf{x}_1, \dots, \mathbf{x}_{n+1}) = \mathbf{x}_1^2 + \cdots + \mathbf{x}_{n+1}^2 - 1 = 0$. In a neighbourhood of the north pole (for example, the northern hemisphere) S^n is given explicitly by the equation $\mathbf{x}_{n+1} = \sqrt{\mathbf{x}_1^2 + \cdots + \mathbf{x}_n^2}$.

If you think of the set, M_3 , of all 3×3 real matrices as \mathbb{R}^9 (because a 3×3 matrix has 9 matrix elements) then

$$SO(3) = \{ A \in M_3 \mid A^t A = \mathbb{1}, \det A = 1 \}$$

is a 3-dimensional surface⁽¹⁾ in \mathbb{R}^9 . $SO(3)$ is the group of all rotations about the origin in \mathbb{R}^3 and is also the set of all orientations of a rigid body,

Example 5 (The Cartesian Product) If M is a manifold of dimension m with atlas \mathfrak{A} and N is a manifold of dimension n with atlas \mathfrak{B} then

$$M \times N = \{ (x, y) \mid x \in M, y \in N \}$$

is an $(m + n)$ -dimensional manifold with atlas

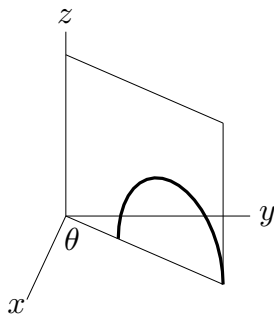
$$\{ (U \times V, \varphi \oplus \psi) \mid (U, \varphi) \in \mathfrak{A}, (V, \psi) \in \mathfrak{B} \} \quad \text{where } \varphi \oplus \psi((x, y)) = (\varphi(x), \psi(y))$$

For example, $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, $S^1 \times \mathbb{R}$ is a cylinder, $S^1 \times S^1$ is a torus (the surface of a donut) and the configuration space of a rigid body is $\mathbb{R}^3 \times SO(3)$ (with the \mathbb{R}^3 components giving the location of the centre of mass of the body and the $SO(3)$ components giving the orientation).

Example 6 (The Torus) The torus T^2 is the two dimensional surface

$$T^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - 1)^2 + z^2 = \frac{1}{4} \}$$

in \mathbb{R}^3 . In cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, the equation of the torus is $(r - 1)^2 + z^2 = \frac{1}{4}$. Fix any θ . Recall that the set of all points in \mathbb{R}^3 that have that value of θ is like one page in an open book. It is a half-plane that contains the z axis. The intersection of the torus with that half plane is a circle of radius $\frac{1}{2}$ centred on $r = 1$, $z = 0$. As φ runs from 0 to 2π , the point $r = 1 + \frac{1}{2} \cos \varphi$, $z = \frac{1}{2} \sin \varphi$,



⁽¹⁾ Note that $A^t A = \mathbb{1}$ forces $\det A \in \{-1, 1\}$. If you fix any $B \in SO(3)$, then, just by continuity, all matrices A that obey $A^t A = \mathbb{1}$ and are sufficiently close to B automatically obey $\det A = 1$. So the equation $\det A = 1$ is redundant. Since $A^t A$ is automatically symmetric, the requirement $A^t A = \mathbb{1}$ gives at most 6 independent equations. In fact they are independent.

$\theta = \theta$ runs over that circle. If we now run θ from 0 to 2π , the circle on the page sweeps out the whole torus. So, as φ runs from 0 to 2π and θ runs from 0 to 2π , the point $(x, y, z) = ((1 + \frac{1}{2} \cos \varphi) \cos \theta, (1 + \frac{1}{2} \cos \varphi) \sin \theta, \frac{1}{2} \sin \varphi)$ runs over the whole torus. So we may build coordinate patches for T^2 using θ and φ (with ranges $(0, 2\pi)$ or $(-\pi, \pi)$) as coordinates.

Example 7 (The Möbius Strip) Take a length of ribbon. Put a half twist in it and glue the ends together. The result is a Möbius Strip. Mathematically, you can think of it as the set $[0, 1] \times (-1, 1)$ but with the points $(0, t)$ and $(1, -t)$ identified (i.e. pretend that they are the same point) for all $-1 < t < 1$. We can view the Möbius Strip as a manifold with the set points $M = [0, 1] \times (-1, 1)$ and the two patch atlas $\mathfrak{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ where

$$U_1 = \left(\frac{1}{8}, \frac{7}{8}\right) \quad \varphi_1(x, y) = (x, y)$$

$$U_2 = \left[0, \frac{1}{4}\right] \cup \left(\frac{3}{4}, 1\right) \quad \varphi_2(x, y) = \begin{cases} (x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ (x - 1, -y) & \text{if } \frac{3}{4} < x < 1 \end{cases}$$

The range of φ_2 is $(-\frac{1}{4}, \frac{1}{4}) \times (-1, 1)$.



Example 8 (Projective n -space, \mathbb{P}^n) The projective n -space, \mathbb{P}^n , is the set of all lines through the origin in \mathbb{R}^{n+1} . If $\vec{x} \in \mathbb{R}^{n+1}$ is nonzero, then there is a unique line $L_{\vec{x}}$ through the origin in \mathbb{R}^{n+1} that contains \vec{x} . Namely $L_{\vec{x}} = \{ \lambda \vec{x} \mid \lambda \in \mathbb{R} \}$. If $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero, then $L_{\vec{x}} = L_{\vec{y}}$ if and only if there is a $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\vec{y} = \lambda \vec{x}$. One choice of atlas for \mathbb{P}^n is $\mathfrak{A} = \{ (U_i, \varphi_i) \mid 1 \leq i \leq n+1 \}$ with

$$U_i = \{ L_{\vec{x}} \mid \vec{x} \in \mathbb{R}^{n+1}, x_i \neq 0 \} \quad \varphi(L_{\vec{x}}) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right) \in \mathbb{R}^n$$

Observe that if φ_i is well-defined, because if $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero and $L_{\vec{x}} = L_{\vec{y}}$, then, for each $1 \leq i \leq n+1$, either both x_i and y_i are zero or both x_i and y_i are nonzero and in the latter case

$$\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right) = \left(\frac{y_1}{y_i}, \dots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \dots, \frac{y_{n+1}}{y_i}\right)$$

Each line through the origin in \mathbb{R}^{n+1} intersects the unit sphere $S^n = \{ \vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1 \}$ in exactly two points and the two points are antipodal (i.e. \vec{x} and $-\vec{x}$). So you can think of \mathbb{P}^n as S^n but with antipodal points identified:

$$\mathbb{P}^{n+1} = \{ \{ \vec{x}, -\vec{x} \} \mid \vec{x} \in S^n \}$$

Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is not horizontal (i.e. with $x_{n+1} \neq 0$) intersects the northern hemisphere $\{ \vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1, x_{n+1} \geq 0 \}$ in exactly one point. Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is horizontal (i.e. with $x_{n+1} = 0$) intersects the northern hemisphere in exactly two points and the two points are antipodal. By ignoring x_{n+1} , you can think of the northern hemisphere as the closed unit disk $\{ \mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1 \}$ in \mathbb{R}^n . So you can think of \mathbb{P}^n as the closed unit ball in \mathbb{R}^n but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified.

In the case of three dimensions, you can also think of $SO(3)$ as being the closed unit disk $\{ \mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| \leq 1 \} \subset \mathbb{R}^3$ but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified. This is because, geometrically, each element of $SO(3)$ is a matrix which implements a rotation by some angle about some axis through the origin in \mathbb{R}^3 . We can associate each $\omega\hat{\Omega} \in \mathbb{R}^3$, where $\hat{\Omega}$ is a unit vector and $\omega \in \mathbb{R}$, with the rotation by an angle $\pi\omega$ about the axis $\hat{\Omega}$. But then any two ω 's that differ by an even integer give the same rotation. So the set of all rotations is associated with $\{ \omega\hat{\Omega} \mid |\omega| \leq 1, \hat{\Omega} \in \mathbb{R}^3, |\hat{\Omega}| = 1 \}$ but with $1\hat{\Omega}$ and $-1\hat{\Omega}$ identified. Thus $SO(3)$ and \mathbb{P}^3 are diffeomorphic.