

# First Order Initial Value Problems

A “first order initial value problem” is the problem of finding a function  $\vec{x}(t)$  which satisfies the conditions

$$\dot{\vec{x}} = \vec{F}(\vec{x}, t) \quad \vec{x}(t_0) = \vec{\xi}_0 \quad (1)$$

where the initial time,  $t_0$ , is a given real number, the initial position,  $\vec{\xi}_0 \in \mathbb{R}^d$ , is a given vector and  $\vec{F} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$  is a given function. We shall assume throughout these notes that  $\vec{F}$  is  $C^\infty$ . By definition, a solution to the initial value problem (1) on the interval  $I$  (which may be open, closed or half-open, but which, of course, contains  $t_0$ ) is a differentiable function  $\vec{x}(t)$  which obeys

$$\vec{x}(t_0) = \vec{\xi}_0 \quad \text{and} \quad \dot{\vec{x}}(t) = \vec{F}(\vec{x}(t), t) \quad \text{for all } t \in I$$

**Remark 1** The restriction to first order derivatives is not significant. Higher order systems can always be converted into first order systems, at the cost of introducing more variables. For example, substituting

$$\vec{y} = \vec{x} \quad \vec{z} = \dot{\vec{x}}$$

into the second order system  $\ddot{\vec{x}} = \vec{F}(\vec{x}, \dot{\vec{x}}, t)$  converts it to the first order system

$$\frac{d}{dt} \begin{bmatrix} \vec{y} \\ \vec{z} \end{bmatrix} = \begin{bmatrix} \vec{z} \\ \vec{F}(\vec{y}, \vec{z}, t) \end{bmatrix}$$

**Theorem 2** Let  $d \in \mathbb{N}$ ,  $t_0 \in \mathbb{R}$ ,  $\vec{\xi}_0 \in \mathbb{R}^d$  and  $I$  be an interval in  $\mathbb{R}$  that contains  $t_0$ . Assume that  $\vec{F} : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$  is  $C^\infty$ .

(a) (Regularity in  $t$ ) A solution to the initial value problem (1) on  $I$  is  $C^\infty$ .

(b) (Uniqueness) If  $\vec{x}(t)$  and  $\vec{y}(t)$  are both solutions to the initial value problem (1) on  $I$ , then  $\vec{x}(t) = \vec{y}(t)$  for all  $t \in I$ .

**Remark:** For uniqueness, it is not sufficient to assume that  $\vec{F}$  is continuous. For example  $x(t) = 0$  and  $x(t) = (\frac{2}{3}t)^{3/2}$  both solve the initial value problem  $x(0) = 0$ ,  $\dot{x} = \sqrt[3]{x}$ .

(c) (Local Existence) Let  $0 < \rho < \infty$  and let  $C_\rho(\vec{\xi}_0)$  denote the closed ball in  $\mathbb{R}^d$  with centre  $\vec{\xi}_0$  and radius  $\rho$ . Then there exists a  $T > 0$  such that the initial value problem (1) has a

solution  $\vec{x}(t)$  on the interval  $[t_0 - T, t_0 + T]$  with  $\vec{x}(t) \in C_\rho(\vec{\xi}_0)$  for all  $t \in [t_0 - T, t_0 + T]$ . Furthermore, if we fix any  $\theta > 0$  and define  $M = \sup \{ |\vec{F}(\vec{x}, t)| \mid \vec{x} \in C_\rho(\vec{\xi}_0), |t - t_0| \leq \theta \}$  (which is the maximum speed that  $\vec{x}$  can have while  $|t - t_0| \leq \theta$  and  $\vec{x}$  is in  $C_\rho(\vec{\xi}_0)$ ), then

$$T \geq \min \left\{ \theta, \frac{\rho}{M} \right\}$$

**Remark:** The bad news is that solutions of (1) need not exist for all  $t$ . For example,  $\dot{x} = x^2$ ,  $x(0) = 1$  has solution  $x(t) = -\frac{1}{t-1}$ , which blows up at  $t = 1$ , despite the fact that the problem “ $\dot{x} = x^2$ ,  $x(0) = 1$ ” shows no sign of pathology. Another example is

$$\begin{aligned} \dot{x} &= x^2 & x(0) &= 1 \\ \dot{y} &= x^2 \cos x - x^3 \sin x & y(0) &= \cos 1 \end{aligned}$$

The solution is  $x(t) = -\frac{1}{t-1}$ ,  $y(t) = -\frac{1}{t-1} \cos \frac{1}{t-1}$ , which oscillates wildly while it is blowing up. The good news is that solutions, which we already know exist at least for times near  $t_0$ , can only fail to globally exist (i.e. exist for all  $t$ ) under specific, known, circumstances, that can often be ruled out.

(d) (Absence of a Global Solution) Let  $-\infty \leq a < b \leq \infty$  with at least one of  $a, b$  finite. Let  $\vec{x}(t)$  be a solution of (1) on the interval  $(a, b)$ . Suppose that there does not exist a solution on any interval that properly contains  $(a, b)$ . Then

$$\begin{aligned} b < \infty &\implies \limsup_{t \rightarrow b^-} |\vec{x}(t)| = \infty \\ a > -\infty &\implies \limsup_{t \rightarrow a^+} |\vec{x}(t)| = \infty \end{aligned}$$

**Remark:** It suffices to consider open intervals  $(a, b)$ , because given a solution on some nonopen interval  $I$ , we can always use part (c) to extend it to a solution on an open interval that contains  $I$ .

(e) (Global Existence) Suppose that a function  $H(\vec{x})$  is conserved by all solutions of (1). In other words, suppose that if  $\vec{x}(t)$  obeys (1), then  $\frac{d}{dt}H(\vec{x}(t)) = 0$ . Suppose further that  $\lim_{|\vec{x}| \rightarrow \infty} |H(\vec{x})| = \infty$ . Then every solution of (1) on an interval properly contained in  $\mathbb{R}$  may be extended to a global solution.

(f) (Regularity with respect to initial conditions and other parameters) Let  $\vec{F}(\vec{x}, t, \vec{\alpha})$  be  $C^\infty$ . Let  $\theta, \rho, a > 0$ . Then there is a  $T > 0$  such that

(i) For each  $\tau, \vec{\zeta}$  and  $\vec{\alpha}$  obeying  $|\tau - t_0| < \theta$ ,  $|\vec{\xi} - \vec{\xi}_0| < \rho$  and  $|\vec{\alpha} - \vec{\alpha}_0| < a$ , there is a solution  $\vec{x}(t; \tau, \vec{\xi}, \vec{\alpha})$  to the initial value problem

$$\dot{\vec{x}} = \vec{F}(\vec{x}, t, \vec{\alpha}) \quad \vec{x}(\tau) = \vec{\xi}$$

on the interval  $[\tau - T, \tau + T]$ .

(ii)  $\vec{x}(t; \tau, \vec{\xi}, \vec{\alpha})$  is  $C^\infty$  as a function of all of its arguments in the region given in (i).

**Proof:** (a) Let  $\vec{x}(t)$  be a solution to the initial value problem (1) on  $I$ . By hypothesis, it is differentiable on  $I$ . Hence it is continuous on  $I$ . As  $\vec{F}$  is continuous,  $\dot{\vec{x}}(t) = \vec{F}(\vec{x}(t), t)$  is a composition of continuous functions, and hence is continuous. That is,  $\vec{x}(t)$  is  $C^1$  on  $I$ . So  $\dot{\vec{x}}(t) = \vec{F}(\vec{x}(t), t)$  is also  $C^1$  with derivative

$$\sum_{j=1}^d \frac{\partial \vec{F}}{\partial x_j}(\vec{x}(t), t) \frac{dx_j}{dt}(t) + \frac{\partial \vec{F}}{\partial t}(\vec{x}(t), t) = \sum_{j=1}^d \frac{\partial \vec{F}}{\partial x_j} F_j(\vec{x}(t), t) + \frac{\partial \vec{F}}{\partial t}(\vec{x}(t), t)$$

which is again continuous. So  $\vec{x}(t)$  is  $C^2$  on  $I$ . And so on.

(b) Let  $J$  be any closed finite interval containing  $t_0$  and contained in  $I$ . We shall show that  $\vec{x}(t) = \vec{y}(t)$  for all  $t \in J$ . Since  $J$  is compact and  $\vec{x}(t)$  and  $\vec{y}(t)$  are continuous, both  $\vec{x}(t)$  and  $\vec{y}(t)$  are bounded for  $t \in J$ . Let  $S$  be a sphere in  $\mathbb{R}^d$  that is sufficiently large that it contains  $\vec{x}(t)$  and  $\vec{y}(t)$  for all  $t \in J$ . Define

$$M_S = \sup \left\{ \sum_{i,j=1}^d \left| \frac{\partial F_i}{\partial x_j}(\vec{x}, t) \right| \mid \vec{x} \in S, t \in J \right\}$$

and, for each  $T > 0$ , let

$$M(T) = \sup \{ |\vec{x}(t) - \vec{y}(t)| \mid |t - t_0| \leq T \}$$

Since  $\vec{x}(t_0) = \vec{y}(t_0)$ ,

$$\begin{aligned} |\vec{x}(t) - \vec{y}(t)| &= \left| \int_{t_0}^t [\dot{\vec{x}}(\tau) - \dot{\vec{y}}(\tau)] d\tau \right| \\ &= \left| \int_{t_0}^t [\vec{F}(\vec{x}(\tau), \tau) - \vec{F}(\vec{y}(\tau), \tau)] d\tau \right| \end{aligned}$$

Now, for any  $\vec{x}, \vec{y} \in S$  and  $\tau \in J$ ,

$$\begin{aligned} |\vec{F}(\vec{x}, \tau) - \vec{F}(\vec{y}, \tau)| &= \left| \int_0^1 \frac{d}{d\sigma} \vec{F}(\sigma \vec{x} + (1 - \sigma) \vec{y}, \tau) d\sigma \right| \\ &= \left| \int_0^1 \sum_{j=1}^d (x_j - y_j) (\partial_{x_j} \vec{F})(\sigma \vec{x} + (1 - \sigma) \vec{y}, \tau) d\sigma \right| \\ &\leq \sum_{j,k=1}^d \int_0^1 |x_j - y_j| |\partial_{x_j} F_k(\sigma \vec{x} + (1 - \sigma) \vec{y}, \tau)| d\sigma \\ &\leq \int_0^1 |\vec{x} - \vec{y}| M_S d\sigma = M_S |\vec{x} - \vec{y}| \end{aligned} \tag{2}$$

Hence, for all  $t \in J$ ,

$$|\vec{x}(t) - \vec{y}(t)| \leq \left| \int_{t_0}^t M_S |\vec{x}(\tau) - \vec{y}(\tau)| d\tau \right|$$

If  $t \in J$  and  $|t - t_0| \leq T$ , the same is true for all  $\tau$  between  $t_0$  and  $t$  so that

$$\begin{aligned} |\vec{x}(t) - \vec{y}(t)| &\leq \left| \int_{t_0}^t M_S |\vec{x}(\tau) - \vec{y}(\tau)| d\tau \right| \leq \left| \int_{t_0}^t M_S M(T) d\tau \right| = |t - t_0| M_S M(T) \\ &\leq T M_S M(T) \end{aligned}$$

Taking the supremum over  $t \in J$  with  $|t - t_0| \leq T$  gives

$$M(T) \leq T M_S M(T)$$

If  $T \leq \frac{1}{2M_S}$ , we have  $M(T) \leq \frac{1}{2}M(T)$  which forces  $M(t) = 0$  for all  $t \in J$  with  $|t - t_0| \leq T$ .

If  $\frac{1}{2M_S}$  is not sufficiently big that  $[t_0 - \frac{1}{2M_S}, t_0 + \frac{1}{2M_S}]$  covers all of  $J$ , repeat with  $t_0$  replaced by  $t_0 \pm \frac{1}{2M_S}$  and then  $t_0 \pm 2 \times \frac{1}{2M_S}$ , and so on.

(c) We shall construct a local solution to the integral equation

$$\vec{x}(t) = \vec{\xi}_0 + \int_{t_0}^t \vec{F}(\vec{x}(\tau), \tau) d\tau \quad (3)$$

which is equivalent to the initial value problem (1). To do so, we construct a sequence of approximate solutions by the algorithm

$$\begin{aligned} \vec{x}^{(0)}(t) &= \vec{\xi}_0 \\ \vec{x}^{(1)}(t) &= \vec{\xi}_0 + \int_{t_0}^t \vec{F}(\vec{x}^{(0)}(\tau), \tau) d\tau \\ \vec{x}^{(2)}(t) &= \vec{\xi}_0 + \int_{t_0}^t \vec{F}(\vec{x}^{(1)}(\tau), \tau) d\tau \\ &\vdots \\ \vec{x}^{(n)}(t) &= \vec{\xi}_0 + \int_{t_0}^t \vec{F}(\vec{x}^{(n-1)}(\tau), \tau) d\tau \end{aligned} \quad (4)$$

If the sequence  $\vec{x}^{(n)}(t)$  of approximate solutions converges uniformly on some closed interval  $[t_0 - T, t_0 + T]$ , then taking the limit of (4) as  $n \rightarrow \infty$  shows that  $\vec{x}(t) = \lim_{n \rightarrow \infty} \vec{x}^{(n)}(t)$  obeys (3) and hence (1).

Define

$$\begin{aligned} T &= \min \left\{ \theta, \frac{\rho}{M} \right\} \\ L &= \sup \left\{ \sum_{i,j=1}^d \left| \frac{\partial F_i}{\partial x_j}(\vec{x}, t) \right| \mid \vec{x} \in C_\rho(\vec{\xi}_0), |t - t_0| \leq T \right\} \\ \mu_k &= \sup \left\{ \left| \vec{x}^{(k)}(t) - \vec{x}^{(k+1)}(t) \right| \mid |t - t_0| \leq T \right\} \end{aligned}$$

It suffices to prove that  $\sum_{k=0}^{\infty} \mu_k < \infty$ , since then we will have uniform convergence.

We first verify that  $\vec{x}^{(n)}(t)$  remains in  $C_\rho(\vec{\xi}_0)$  for all  $n$  and  $|t - t_0| \leq T$ . Since  $MT \leq \rho$ , we have

$$\begin{aligned} \sup_{|t-t_0| \leq T} |\vec{x}^{(n)}(t) - \vec{\xi}_0| &\leq \sup_{|t-t_0| \leq T} \left| \int_{t_0}^t \vec{F}(\vec{x}^{(n-1)}(\tau), \tau) d\tau \right| \\ &\leq T \sup_{|\tau-t_0| \leq T} |\vec{F}(\vec{x}^{(n-1)}(\tau), \tau)| \\ &\leq TM \leq \rho \quad \text{if } \vec{x}^{(n-1)}(\tau) \in C_\rho(\vec{\xi}_0) \text{ for all } |\tau - t_0| \leq T \end{aligned}$$

and it follows, by induction on  $n$ , that  $|\vec{x}^{(n)}(t) - \vec{\xi}_0| \leq \rho$  (i.e.  $\vec{x}^{(n)}(t) \in C_\rho(\vec{\xi}_0)$ ) for all  $|t - t_0| \leq T$ .

Now

$$\begin{aligned} |\vec{x}^{(k)}(t) - \vec{x}^{(k+1)}(t)| &= \left| \int_{t_0}^t [\vec{F}(\vec{x}^{(k-1)}(\tau), \tau) - \vec{F}(\vec{x}^{(k)}(\tau), \tau)] d\tau \right| \\ &\leq L \left| \int_{t_0}^t |\vec{x}^{(k-1)}(\tau_1) - \vec{x}^{(k)}(\tau_1)| d\tau_1 \right| \end{aligned}$$

as in (2). Iterating this bound gives

$$\begin{aligned} |\vec{x}^{(k)}(t) - \vec{x}^{(k+1)}(t)| &\leq L^2 \left| \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 |\vec{x}^{(k-2)}(\tau_2) - \vec{x}^{(k-1)}(\tau_2)| \right| \\ &\quad \vdots \\ &\leq L^k \left| \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k |\vec{x}^{(0)}(\tau_k) - \vec{x}^{(1)}(\tau_k)| \right| \\ &= L^k \left| \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k |\vec{\xi}_0 - \vec{x}^{(1)}(\tau_k)| \right| \\ &\leq L^k \left| \int_{t_0}^t d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k \rho \right| \\ &= L^k \rho \frac{1}{k!} |t - t_0|^k \end{aligned}$$

and hence

$$\sum_{k=0}^{\infty} \mu_k \leq \sum_{k=0}^{\infty} \rho \frac{1}{k!} (LT)^k = \rho e^{LT}$$

(d) This is easy. If the solution stays bounded near one end of the interval, say on  $[b - \varepsilon, b)$  with  $b < \infty$ , we can use the local existence result of part (c) to extend the solution past  $b$  simply by choosing  $t_0$  very close to  $b$ .

(e) This is also easy. If  $h = H(\vec{\xi}_0)$ , then any solution  $\vec{x}(t)$  must remain in

$$X_h = \{ \vec{x} \mid H(\vec{x}) = h \}$$

Since  $\lim_{|\vec{x}| \rightarrow \infty} |H(\vec{x})| = \infty$ , the region  $X_h$  is bounded. Apply part (d).

(f) (Outline of proof only.) The idea of the proof is the same as that for part (c) (local existence). For each fixed  $(\tau, \vec{\xi}, \vec{\alpha})$ , the solution  $\vec{x}(t; \tau, \vec{\xi}, \vec{\alpha})$  is the limit of the sequence defined recursively by

$$\begin{aligned} \vec{x}^{(0)}(t; \tau, \vec{\xi}, \vec{\alpha}) &= \vec{\xi} \\ \vec{x}^{(n)}(t; \tau, \vec{\xi}, \vec{\alpha}) &= \vec{\xi} + \int_{\tau}^t \vec{F}(\vec{x}^{(n-1)}(\tau'; \tau, \vec{\xi}, \vec{\alpha}), \tau', \vec{\alpha}) d\tau' \quad n = 1, 2, 3, \dots \end{aligned} \quad (5)$$

As in part (a), it is obvious by induction on  $k$  that  $\vec{x}^{(k)}(t; \tau, \vec{\xi}, \vec{\alpha})$  is  $C^\infty$  in all of its arguments. To prove that the limit as  $k \rightarrow \infty$  is  $C^\infty$ , it suffices to prove that all derivatives of  $\vec{x}^{(k)}$  converge uniformly as  $k \rightarrow \infty$ . So it suffices to prove that for any (possibly higher order) derivative  $D$ ,

$$\mu_{D,k} = \sup |D\vec{x}^{(k)} - D\vec{x}^{(k+1)}|$$

obeys  $\sum_{k=0}^{\infty} \mu_{D,k} < \infty$ . But we can bound  $\mu_{D,k}$  by applying  $D$  to the recursion relation (5) and then using the method of part (c). ■