First Order Initial Value Problems

A "first order initial value problem" is the problem of finding a function $\vec{x}(t)$ which satisfies the conditions

$$\dot{\vec{x}} = \vec{F}(\vec{x}, t) \qquad \vec{x}(t_0) = \vec{\xi}_0 \tag{1}$$

where the initial time, t_0, is a given real number, the initial position, $\vec{\xi}_0 \in \mathbb{R}^d$, is a given vector and $\vec{F} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a given function. We shall assume throughout these notes that $\vec{F}$ is C^∞. By definition, a solution to the initial value problem (1) on the interval I (which may be open, closed or half–open, but which, of course, contains t_0) is a differentiable function $\vec{x}(t)$ which obeys

$$\vec{x}(t_0) = \vec{\xi}_0 \qquad \text{and} \qquad \dot{\vec{x}}(t) = \vec{F}\big(\vec{x}(t), t\big) \quad \text{for all } t \in I$$

Remark 1 The restriction to first order derivatives is not significant. Higher order systems can always be converted into first order systems, at the cost of introducing more variables. For example, substituting

$$\vec{y} = \vec{x} \qquad \vec{z} = \dot{\vec{x}}$$

into the second order system $\ddot{\vec{x}} = \vec{F}(\vec{x}, \dot{\vec{x}}, t)$ converts it to the first order system

$$\frac{d}{dt}\begin{bmatrix}\vec{y}\\ \vec{z}\end{bmatrix} = \begin{bmatrix}\vec{z}\\ \vec{F}(\vec{y}, \vec{z}, t)\end{bmatrix}$$

Theorem 2 *Let $d \in \mathbb{N}$, $t_0 \in \mathbb{R}$, $\vec{\xi}_0 \in \mathbb{R}^d$ and I be an interval in $\mathbb{R}$ that contains t_0. Assume that $\vec{F} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is C^∞.*

(a) (Regularity in t) A solution to the initial value problem (1) on I is C^∞.

(b) (Uniqueness) If $\vec{x}(t)$ and $\vec{y}(t)$ are both solutions to the initial value problem (1) on I, then $\vec{x}(t) = \vec{y}(t)$ for all $t \in I$.

Remark*: For uniqueness, it is* not *sufficient to assume that $\vec{F}$ is continuous. For example $x(t) = 0$ and $x(t) = \left(\frac{2}{3}t\right)^{3/2}$ both solve the initial value problem $x(0) = 0$, $\dot{x} = \sqrt[3]{x}$.*

(c) (Local Existence) Let $0 < \rho < \infty$ and let $C_\rho(\vec{\xi}_0)$ denote the closed ball in $\mathbb{R}^d$ with centre ξ_0 and radius ρ. Then there exists a $T > 0$ such that the initial value problem (1) has a

solution $\vec{x}(t)$ on the interval $[t_0 - T, t_0 + T]$ with $\vec{x}(t) \in C_\rho(\vec{\xi}_0)$ for all $t \in [t_0 - T, t_0 + T]$. Furthermore, if we fix any $\theta > 0$ and define $M = \sup\left\{ |\vec{F}(\vec{x}, t)| \;\middle|\; \vec{x} \in C_\rho(\vec{\xi}_0),\ |t - t_0| \le \theta \right\}$ (which is the maximum speed that $\vec{x}$ can have while $|t - t_0| \le \theta$ and $\vec{x}$ is in $C_\rho(\vec{\xi}_0)$), then

$$T \ge \min\left\{\theta, \tfrac{\rho}{M}\right\}$$

Remark*: The bad news is that solutions of (1) need not exist for all t. For example, $\dot{x} = x^2$, $x(0) = 1$ has solution $x(t) = -\frac{1}{t-1}$, which blows up at $t = 1$, despite the fact that the problem "$\dot{x} = x^2$, $x(0) = 1$" shows no sign of pathology. Another example is*

$$\begin{aligned} \dot{x} &= x^2 & x(0) &= 1 \\ \dot{y} &= x^2 \cos x - x^3 \sin x & y(0) &= \cos 1 \end{aligned}$$

The solution is $x(t) = -\frac{1}{t-1}$, $y(t) = -\frac{1}{t-1} \cos \frac{1}{t-1}$, which oscillates wildly while it is blowing up. The good news is that solutions, which we already know exist at least for times near t_0, can only fail to globally exist (i.e. exist for all t) under specific, known, circumstances, that can often be ruled out.

(d) (Absence of a Global Solution) Let $-\infty \le a < b \le \infty$ with at least one of a, b finite. Let $\vec{x}(t)$ be a solution of (1) on the interval (a, b). Suppose that there does not exist a solution on any interval that properly contains (a, b). Then

$$\begin{aligned} b < \infty &\implies \limsup_{t \to b-} |\vec{x}(t)| = \infty \\ a > -\infty &\implies \limsup_{t \to a+} |\vec{x}(t)| = \infty \end{aligned}$$

Remark*: It suffices to consider open intervals (a, b), because given a solution on some nonopen interval I, we can always use part (c) to extend it to a solution on an open interval that contains I.*

(e) (Global Existence) Suppose that a function $H(\vec{x})$ is conserved by all solutions of (1). In other words, suppose that if $\vec{x}(t)$ obeys (1), then $\frac{d}{dt} H(\vec{x}(t)) = 0$. Suppose further that $\lim_{|\vec{x}| \to \infty} \big|H(\vec{x})\big| = \infty$. Then every solution of (1) on an interval properly contained in $\mathbb{R}$ may be extended to a global solution.

(f) (Regularity with respect to initial conditions and other parameters) Let $\vec{F}(\vec{x}, t, \vec{\alpha})$ be C^∞. Let $\theta, \rho, a > 0$. Then there is a $T > 0$ such that

- *(i) For each $\tau, \vec{\zeta}$ and $\vec{\alpha}$ obeying $|\tau - t_0| < \theta$, $|\vec{\xi} - \vec{\xi}_0| < \rho$ and $|\vec{\alpha} - \vec{\alpha}_0| < a$, there is a solution $\vec{x}(t\,; \tau, \vec{\xi}, \vec{\alpha})$ to the initial value problem*

$$\dot{\vec{x}} = \vec{F}(\vec{x}, t, \vec{\alpha}) \qquad \vec{x}(\tau) = \vec{\xi}$$

  on the interval $[\tau - T, \tau + T]$.
- *(ii) $\vec{x}(t\,; \tau, \vec{\xi}, \vec{\alpha})$ is C^∞ as a function of all of its arguments in the region given in (i).*

Proof: (a) Let $\vec{x}(t)$ be a solution to the initial value problem (1) on I. By hypothesis, it is differentiable on I. Hence it is continuous on I. As $\vec{F}$ is continuous, $\dot{\vec{x}}(t) = \vec{F}(\vec{x}(t), t)$ is a composition of continuous functions, and hence is continuous. That is, $\vec{x}(t)$ is C^1 on I. So $\dot{\vec{x}}(t) = \vec{F}(\vec{x}(t), t)$ is also C^1 with derivative

$$\sum_{j=1}^{d} \tfrac{\partial \vec{F}}{\partial x_j}(\vec{x}(t), t)\tfrac{dx_j}{dt}(t) + \tfrac{\partial \vec{F}}{\partial t}(\vec{x}(t), t) = \sum_{j=1}^{d} \tfrac{\partial \vec{F}}{\partial x_j}\, F_j(\vec{x}(t), t) + \tfrac{\partial \vec{F}}{\partial t}(\vec{x}(t), t)$$

which is again continuous. So $\vec{x}(t)$ is C^2 on I. And so on.

(b) Let J be any closed finite interval containing t_0 and contained in I. We shall show that $\vec{x}(t) = \vec{y}(t)$ for all $t \in J$. Since J is compact and $\vec{x}(t)$ and $\vec{y}(t)$ are continuous, both $\vec{x}(t)$ and $\vec{y}(t)$ are bounded for $t \in J$. Let S be a sphere in $\mathbb{R}^d$ that is sufficiently large that it contains $\vec{x}(t)$ and $\vec{y}(t)$ for all $t \in J$. Define

$$M_S = \sup\Big\{ \sum_{i,j=1}^{d} \big|\tfrac{\partial F_i}{\partial x_j}(\vec{x}, t)\big| \;\Big|\; \vec{x} \in S,\ t \in J \Big\}$$

and, for each $T > 0$, let

$$M(T) = \sup\big\{ |\vec{x}(t) - \vec{y}(t)| \;\big|\; |t - t_0| \le T \big\}$$

Since $\vec{x}(t_0) = \vec{y}(t_0)$,

$$\begin{aligned}
\big|\vec{x}(t) - \vec{y}(t)\big| &= \left| \int_{t_0}^{t} \big[\dot{\vec{x}}(\tau) - \dot{\vec{y}}(\tau)\big]\, d\tau \right| \\
&= \left| \int_{t_0}^{t} \big[\vec{F}\big(\vec{x}(\tau), \tau\big) - \vec{F}\big(\vec{y}(\tau), \tau\big)\big]\, d\tau \right|
\end{aligned}$$

Now, for any $\vec{x}, \vec{y} \in S$ and $\tau \in J$,

$$\begin{aligned}
\big|\vec{F}\big(\vec{x}, \tau\big) - \vec{F}\big(\vec{y}, \tau\big)\big| &= \left| \int_0^1 \tfrac{d}{d\sigma} \vec{F}\big(\sigma\vec{x} + (1-\sigma)\vec{y}, \tau\big)\, d\sigma \right| \\
&= \left| \int_0^1 \sum_{j=1}^{d} (x_j - y_j)\, (\partial_{x_j}\vec{F})\big(\sigma\vec{x} + (1-\sigma)\vec{y}, \tau\big)\, d\sigma \right| \\
&\le \sum_{j,k=1}^{d} \int_0^1 |x_j - y_j|\, \big|\partial_{x_j} F_k\big(\sigma\vec{x} + (1-\sigma)\vec{y}, \tau\big)\big|\, d\sigma \\
&\le \int_0^1 |\vec{x} - \vec{y}|\ M_S\, d\sigma = M_S|\vec{x} - \vec{y}|
\end{aligned} \tag{2}$$

Hence, for all $t \in J$,

$$\big|\vec{x}(t) - \vec{y}(t)\big| \le \left| \int_{t_0}^{t} M_S \, |\vec{x}(\tau) - \vec{y}(\tau)| \; d\tau \right|$$

If $t \in J$ and $|t - t_0| \le T$, the same is true for all τ between t_0 and t so that

$$\begin{aligned} \big|\vec{x}(t) - \vec{y}(t)\big| &\le \left| \int_{t_0}^{t} M_S \, |\vec{x}(\tau) - \vec{y}(\tau)| \; d\tau \right| \le \left| \int_{t_0}^{t} M_S \, M(T) \; d\tau \right| = |t - t_0| M_S \, M(T) \\ &\le T M_S M(T) \end{aligned}$$

Taking the supremum over $t \in J$ with $|t - t_0| \le T$ gives

$$M(T) \le T M_S M(T)$$

If $T \le \frac{1}{2M_S}$, we have $M(T) \le \frac{1}{2} M(T)$ which forces $M(t) = 0$ for all $t \in J$ with $|t - t_0| \le T$. If $\frac{1}{2M_S}$ is not sufficiently big that $[t_0 - \frac{1}{2M_S}, t_0 + \frac{1}{2M_S}]$ covers all of J, repeat with t_0 replaced by $t_0 \pm \frac{1}{2M_S}$ and then $t_0 \pm 2 \times \frac{1}{2M_S}$, and so on.

(c) We shall construct a local solution to the integral equation

$$\vec{x}(t) = \vec{\xi}_0 + \int_{t_0}^{t} \vec{F}\big(\vec{x}(\tau), \tau\big) \; d\tau \tag{3}$$

which is equivalent to the initial value problem (1). To do so, we construct a sequence of approximate solutions by the algorithm

$$\begin{aligned} \vec{x}^{(0)}(t) &= \vec{\xi}_0 \\ \vec{x}^{(1)}(t) &= \vec{\xi}_0 + \int_{t_0}^{t} \vec{F}\big(\vec{x}^{(0)}(\tau), \tau\big) \; d\tau \\ \vec{x}^{(2)}(t) &= \vec{\xi}_0 + \int_{t_0}^{t} \vec{F}\big(\vec{x}^{(1)}(\tau), \tau\big) \; d\tau \\ &\vdots \\ \vec{x}^{(n)}(t) &= \vec{\xi}_0 + \int_{t_0}^{t} \vec{F}\big(\vec{x}^{(n-1)}(\tau), \tau\big) \; d\tau \end{aligned} \tag{4}$$

If the sequence $\vec{x}^{(n)}(t)$ of approximate solutions converges uniformly on some closed interval $[t_0 - T, t_0 + T]$, then taking the limit of (4) as $n \to \infty$ shows that $\vec{x}(t) = \lim_{n\to\infty} \vec{x}^{(n)}(t)$ obeys (3) and hence (1).

Define

$$\begin{aligned} T &= \min\big\{\theta, \tfrac{\rho}{M}\big\} \\ L &= \sup\Big\{ \sum_{i,j=1}^{d} \big|\tfrac{\partial F_i}{\partial x_j}(\vec{x}, t)\big| \;\Big|\; \vec{x} \in C_\rho(\vec{\xi}_0),\ |t - t_0| \le T \Big\} \\ \mu_k &= \sup\Big\{ \big|\vec{x}^{(k)}(t) - \vec{x}^{(k+1)}(t)\big| \;\Big|\; |t - t_0| \le T \Big\} \end{aligned}$$

It suffices to prove that $\sum_{k=0}^{\infty} \mu_k < \infty$, since then we will have uniform convergence.

We first verify that $\vec{x}^{(n)}(t)$ remains in $C_\rho(\vec{\xi}_0)$ for all n and $|t-t_0| \le T$. Since $MT \le \rho$, we have

$$\begin{aligned}
\sup_{|t-t_0|\le T} \big|\vec{x}^{(n)}(t) - \vec{\xi}_0\big| &\le \sup_{|t-t_0|\le T} \left| \int_{t_0}^{t} \vec{F}\big(\vec{x}^{(n-1)}(\tau), \tau\big)\ d\tau \right| \\
&\le T \sup_{|\tau-t_0|\le T} \big|\vec{F}\big(\vec{x}^{(n-1)}(\tau), \tau\big)\big| \\
&\le TM \le \rho \qquad \text{if } \vec{x}^{(n-1)}(\tau) \in C_\rho(\vec{\xi}_0) \text{ for all } |\tau - t_0| \le T
\end{aligned}$$

and it follows, by induction on n, that $\big|\vec{x}^{(n)}(t) - \vec{\xi}_0\big| \le \rho$ (i.e. $\vec{x}^{(n)}(t) \in C_\rho(\vec{\xi}_0)$) for all $|t - t_0| \le T$.

Now

$$\begin{aligned}
\big|\vec{x}^{(k)}(t) - \vec{x}^{(k+1)}(t)\big| &= \left| \int_{t_0}^{t} \big[\vec{F}\big(\vec{x}^{(k-1)}(\tau), \tau\big) - \vec{F}\big(\vec{x}^{(k)}(\tau), \tau\big)\big]\ d\tau \right| \\
&\le L \left| \int_{t_0}^{t} \big|\vec{x}^{(k-1)}(\tau_1) - \vec{x}^{(k)}(\tau_1)\big|\ d\tau_1 \right|
\end{aligned}$$

as in (2). Iterating this bound gives

$$\begin{aligned}
\big|\vec{x}^{(k)}(t) - \vec{x}^{(k+1)}(t)\big| &\le L^2 \left| \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2\ \big|\vec{x}^{(k-2)}(\tau_2) - \vec{x}^{(k-1)}(\tau_2)\big| \right| \\
&\ \ \vdots \\
&\le L^k \left| \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2\ \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k\ \big|\vec{x}^{(0)}(\tau_k) - \vec{x}^{(1)}(\tau_k)\big| \right| \\
&= L^k \left| \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2\ \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k\ \big|\vec{\xi}_0 - \vec{x}^{(1)}(\tau_k)\big| \right| \\
&\le L^k \left| \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2\ \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k\ \rho \right| \\
&= L^k \rho \tfrac{1}{k!} |t - t_0|^k
\end{aligned}$$

and hence

$$\sum_{k=0}^{\infty} \mu_k \le \sum_{k=0}^{\infty} \rho \tfrac{1}{k!} (LT)^k = \rho e^{LT}$$

(d) This is easy. If the solution stays bounded near one end of the interval, say on $[b-\varepsilon, b)$ with $b < \infty$, we can use the local existence result of part (c) to extend the solution past b simply by choosing t_0 very close to b.

(e) This is also easy. If $h = H(\vec{\xi}_0)$, then any solution $\vec{x}(t)$ must remain in

$$X_h = \left\{ \ \vec{x} \ \middle| \ H(\vec{x}) = h \ \right\}$$

Since $\lim\limits_{|\vec{x}|\to\infty} \big|H(\vec{x})\big| = \infty$, the region X_h is bounded. Apply part (d).

(f) (Outline of proof only.) The idea of the proof is the same as that for part (c) (local existence). For each fixed $(\tau, \vec{\xi}, \vec{\alpha})$, the solution $\vec{x}(t\,;\,\tau, \vec{\xi}, \vec{\alpha})$ is the limit of the sequence defined recursively by

$$\begin{aligned}
\vec{x}^{(0)}(t\,;\,\tau, \vec{\xi}, \vec{\alpha}) &= \vec{\xi} \\
\vec{x}^{(n)}(t\,;\,\tau, \vec{\xi}, \vec{\alpha}) &= \vec{\xi} + \int_\tau^t \vec{F}\big(\vec{x}^{(n-1)}(\tau'\,;\,\tau, \vec{\xi}, \vec{\alpha}), \tau', \vec{\alpha}\big)\ d\tau' \qquad n = 1,\ 2,\ 3,\ \cdots
\end{aligned} \tag{5}$$

As in part (a), it is obvious by induction on k that $\vec{x}^{(k)}(t\,;\,\tau, \vec{\xi}, \vec{\alpha})$ is C^∞ in all of its arguments. To prove that the limit as $k \to \infty$ is C^∞, it suffices to prove that all derivatives of $\vec{x}^{(k)}$ converge uniformly as $k \to \infty$. So it suffices to prove that for any (possibly higher order) derivative D,

$$\mu_{D,k} = \sup \big| D\vec{x}^{(k)} - D\vec{x}^{(k+1)} \big|$$

obeys $\sum\limits_{k=0}^{\infty} \mu_{D,k} < \infty$. But we can bound $\mu_{D,k}$ by applying D to the recursion relation (5) and then using the method of part (c). ■