Euler Angles

Euler angles are three angles that provide coordinates on $SO(3)$. Denote by $R_1(\theta)$ and $R_3(\theta)$ rotations about the $x-$ and $z-$ axes, respectively, by an angle θ. That is

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each $(\varphi, \theta, \psi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$, the matrix $R_3(\varphi)R_1(\theta)R_3(\psi)$ is certainly in $SO(3)$. The following theorem says that each $R \in SO(3)$ has a representation of the form $R_3(\varphi)R_1(\theta)R_3(\psi)$ with $(\varphi, \theta, \psi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ and that this representation is unique unless θ is 0 or π (in which case $R_3(\varphi)R_1(\theta)R_3(\psi)$ depends only on $\varphi+\psi$ or $\varphi-\psi$, respectively).

Theorem.

(a) (Existence) For each $R \in SO(3)$, there an $(\varphi, \theta, \psi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ such that $R = R_3(\varphi)R_1(\theta)R_3(\psi)$.

(b) (Uniqueness) Let $(\varphi, \theta, \psi), (\varphi', \theta', \psi') \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ and assume that $R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi')R_1(\theta')R_3(\psi')$. Then $\theta = \theta'$. If $\theta \neq 0, \pi$, then $\varphi = \varphi'$ and $\psi = \psi'$.

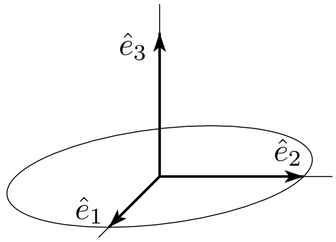
(c) If $R = \begin{bmatrix} \hat\imath_1 & \hat\imath_2 & \hat\imath_3 \end{bmatrix}$, then

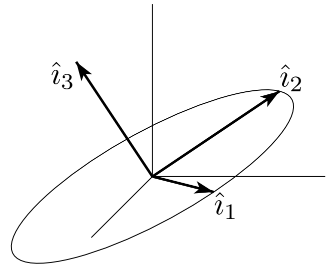
- *θ is the angle between $\hat\imath_3$ and the z–axis and*
- *if $\theta \notin \{0, \pi\}$, then φ is the angle between the projection of $\hat\imath_3$ on the xy–plane and the x–axis, plus $\frac{\pi}{2}$*

Proof: (a) Write $R = \begin{bmatrix} \hat\imath_1 & \hat\imath_2 & \hat\imath_3 \end{bmatrix}$ where $\hat\imath_1$, $\hat\imath_2$ and $\hat\imath_3$ are three mutually perpendicular unit vectors that form a right handed triple and use

$$\hat e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

to denote the standard basis for $\mathbb{R}^3$.



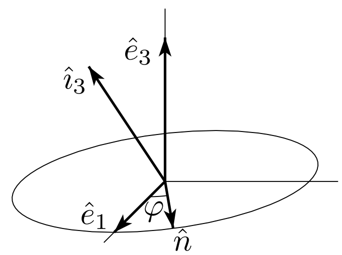


Set

$$\hat n = \begin{cases} \frac{\hat e_3\times\hat\imath_3}{|\hat e_3\times\hat\imath_3|} & \text{if } \hat\imath_3\neq\pm\hat e_3 \\ \hat\imath_1 & \text{if } \hat\imath_3=\hat e_3 \\ -\hat\imath_1 & \text{if } \hat\imath_3=-\hat e_3 \end{cases}$$

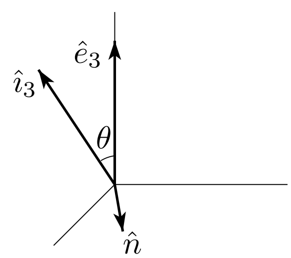
Since $\hat e_1$ and $\hat n$ are both unit vectors that are perpendicular to $\hat e_3$, there is an angle $\varphi\in[0,2\pi)$ such that rotation about $\hat e_3$ by φ maps $\hat e_1$ to $\hat n$. This rotation leaves $\hat e_3$ invariant. Thus

$$\hat n = R_3(\varphi)\hat e_1 \qquad \hat e_3 = R_3(\varphi)\hat e_3$$



Now $\hat e_3$ and $\hat\imath_3$ are both unit vectors that are perpendicular to $\hat n$. So there is an angle $\theta\in[0,2\pi)$ such that rotation about $\hat n$ by θ maps $\hat e_3$ to $\hat\imath_3$. In fact $\theta\in[0,\pi]$ because[(1)] $\hat n$ has direction $\hat e_3\times\hat\imath_3$. The rotation about $\hat n$ by θ is implemented by $R_3(\varphi)R_1(\theta)R_3(-\varphi)$ (since $\frac{d}{d\theta}R_3(\varphi)R_1(\theta)R_3(-\varphi)\vec x = R_3(\varphi)\big(\hat e_1\times(R_3(-\varphi)\vec x)\big) = \big(R_3(\varphi)\hat e_1\big)\times\vec x = \hat n\times\vec x$). So

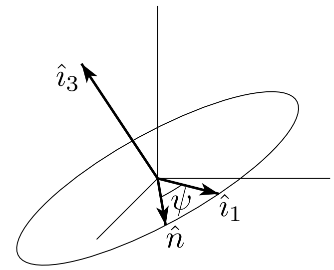
$$\begin{aligned} \hat\imath_3 &= R_3(\varphi)R_1(\theta)R_3(-\varphi)\hat e_3 = R_3(\varphi)R_1(\theta)\hat e_3 \\ \hat n &= R_3(\varphi)R_1(\theta)R_3(-\varphi)\hat n = R_3(\varphi)R_1(\theta)\hat e_1 \end{aligned}$$



Finally, $\hat\imath_1$ and $\hat n$ are both unit vectors that are perpendicular to $\hat\imath_3$. So there is an angle $\psi\in[0,2\pi)$ such that rotation about $\hat\imath_3$ by ψ maps $\hat n$ to $\hat\imath_1$. The rotation about $\hat\imath_3$ by ψ is implemented by $R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)$. (Note that $R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)\hat\imath_3=\hat\imath_3$.) So

(1) Use a coordinate system with $\hat e_3$ on the positive z–axis and $\hat n$ on the positive y–axis. If $\hat\imath_3 = a\hat e_1+b\hat e_2+c\hat e_3$, then $\hat e_3\times\hat\imath_3 = a\hat e_2-b\hat e_1$. For this to be a positive multiple of $\hat e_2$, we need $b=0$ and $a>0$. So $\hat\imath_3$ must be in the xz–plane with positive x–component.

$$\hat\imath_1 = R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)\hat n = R_3(\varphi)R_1(\theta)R_3(\psi)\hat e_1$$
$$\hat\imath_3 = R_3(\varphi)R_1(\theta)R_3(\psi)R_1(-\theta)R_3(-\varphi)\hat\imath_3 = R_3(\varphi)R_1(\theta)R_3(\psi)\hat e_3$$



In other words, the two matrices R and $R_3(\varphi)R_1(\theta)R_3(\psi)$ have the same first column (namely $\hat\imath_1$) and the same third column (namely $\hat\imath_3$). The columns of any matrix in $SO(3)$ form a right handed triple of mutually perpendicular unit vectors. As both R and $R_3(\varphi)R_1(\theta)R_3(\psi)$ are in $SO(3)$, they both must have the same middle column too (namely $\hat\imath_2$).

Uniqueness: Let $(\varphi,\theta,\psi),(\varphi',\theta',\psi')\in[0,2\pi)\times[0,\pi]\times[0,2\pi)$ and assume that

$$R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi')R_1(\theta')R_3(\psi')$$

Multiplying on the left by $R_3(-\varphi')$ and on the right by $R_3(-\psi')$ gives

$$R_3(\varphi-\varphi')R_1(\theta)R_3(\psi-\psi') = R_1(\theta')$$

In particular $R_3(\varphi-\varphi')R_1(\theta)R_3(\psi-\psi')\hat e_3 = R_1(\theta')\hat e_3$. As

$$R_1(\theta')\hat e_3 = \begin{bmatrix} 0 \\ -\sin\theta' \\ \cos\theta' \end{bmatrix}$$

and

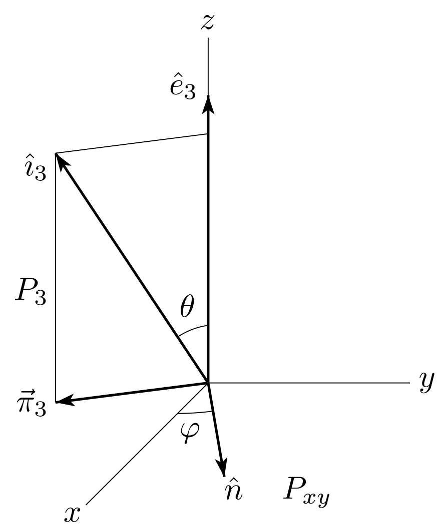
$$\begin{aligned} R_3(\varphi-\varphi')R_1(\theta)R_3(\psi-\psi')\hat e_3 &= R_3(\varphi-\varphi')R_1(\theta)\hat e_3 \\ &= \begin{bmatrix} \cos(\varphi-\varphi') & -\sin(\varphi-\varphi') & 0 \\ \sin(\varphi-\varphi') & \cos(\varphi-\varphi') & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \sin(\varphi-\varphi')\sin\theta \\ -\cos(\varphi-\varphi')\sin\theta \\ \cos\theta \end{bmatrix} \end{aligned}$$

we have

$$\begin{bmatrix} 0 \\ -\sin\theta' \\ \cos\theta' \end{bmatrix} = \begin{bmatrix} \sin(\varphi-\varphi')\sin\theta \\ -\cos(\varphi-\varphi')\sin\theta \\ \cos\theta \end{bmatrix}$$

As $0\le\theta,\theta'\in[0,\pi]$ and cos is 1–1 on $[0\pi]$, we now have that $\theta=\theta'$. As long as $\theta\ne 0,\pi$, we also have $\sin(\varphi-\varphi')=0$ and $\cos(\varphi-\varphi')=1$, which forces $\varphi-\varphi'$ to be an integer multiple of 2π. Since $0\le\varphi,\varphi'<2\pi$, this in turn forces $\varphi=\varphi'$. Finally, $R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi')R_1(\theta')R_3(\psi')$ has now reduced to $R_3(\varphi)R_1(\theta)R_3(\psi) = R_3(\varphi)R_1(\theta)R_3(\psi')$, which is equivalent to $R_3(\psi)=R_3(\psi')$. Since $0\le\psi,\psi'<2\pi$, this forces $\psi=\psi'$.

(c) Assume that $\theta \neq 0$. Then $\hat{n} = \frac{\hat{e}_3 \times \hat{\imath}_3}{|\hat{e}_3 \times \hat{\imath}_3|}$ is perpendicular to both $\hat{\imath}_3$ and $\hat{e}_3$. We defined θ to be the angle of rotation about $\hat{n}$ that maps $\hat{e}_3$ to $\hat{\imath}_3$. By definition, this is the angle between $\hat{e}_3$ and $\hat{\imath}_3$. Let P_{xy} denote the xy–plane and P_3 denote the plane containing $\hat{e}_3$ and $\hat{\imath}_3$. Then $\hat{n}$ lies in P_{xy} (because it is perpendicular to $\hat{e}_3$) and is perpendicular to P_3. The projection of $\hat{\imath}_3$ onto the xy–plane (denoted $\vec{\pi}_3$ in the figure below) lies in both P_3 and P_{xy} and so is an angle $\frac{\pi}{2}$ from $\hat{n}$. We defined φ to be the angle between $\hat{e}_1$ and $\hat{n}$.



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