Duhamel’s Formula

**Theorem (Duhamel)** Let $[A_{i,j}(t)]_{1 \leq i,j \leq n}$ be a matrix-valued function of $t \in \mathbb{R}$ that is $C^\infty$ in the sense that each matrix element $A_{i,j}(t)$ is $C^\infty$. Then

$$\frac{d}{dt}e^{A(t)} = \int_0^1 e^{sA(t)}A'(t)e^{(1-s)A(t)} \, ds$$

**Proof:** We first use Taylor’s formula with remainder, applied separately to each matrix element, to give

$$A(t + h) = A(t) + A'(t)h + \int_t^{t+h} (t + h - \tau)A''(\tau) \, d\tau$$

$$= A(t) + A'(t)h + h^2 \int_0^1 (1 - x)A''(t + hx) \, dx \quad \text{where } \tau = t + hx$$

$$= A(t) + A'(t)h + B(t, h)h^2 \quad \text{where } B(t, h) = \int_0^1 (1 - x)A''(t + hx) \, dx$$

Observe that $B(t, h)$ is $C^\infty$ in $t$ and $h$. Define

$$E(s) = e^{sA(t+h)}e^{(1-s)A(t)}$$

Then

$$e^{A(t+h)} - e^{A(t)} = E(1) - E(0) = \int_0^1 E'(s) \, ds$$

$$= \int_0^1 \{ e^{sA(t+h)}A(t + h)e^{(1-s)A(t)} - e^{sA(t+h)}A(t)e^{(1-s)A(t)} \} \, ds$$

In computing $E'(s)$ we used the product rule and the fact that, for any constant square matrix $C$, $\frac{d}{ds}e^{sC} = Ce^{sC} = e^{sC}C$. (This is easily proven by expanding the exponentials in power series.) Continuing the computation,

$$\frac{1}{h}\left[ e^{A(t+h)} - e^{A(t)} \right] = \int_0^1 e^{sA(t+h)}\frac{1}{h}[A(t + h) - A(t)]e^{(1-s)A(t)} \, ds$$

$$= \int_0^1 e^{sA(t+h)}[A'(t) + B(t, h)h]e^{(1-s)A(t)} \, ds$$

It now suffices to take the limit $h \to 0$.  

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