Duhamel's Formula

Theorem (Duhamel) *Let $\big[A_{i,j}(t)\big]_{1\le i,j\le n}$ be a matrix–valued function of $t\in\mathbb{R}$ that is C^∞ in the sense that each matrix element $A_{i,j}(t)$ is C^∞. Then*

$$\tfrac{d}{dt}e^{A(t)} = \int_0^1 e^{sA(t)}A'(t)e^{(1-s)A(t)}\ ds$$

Proof: We first use Taylor's formula with remainder, applied separately to each matrix element, to give

$$\begin{aligned} A(t+h) &= A(t) + A'(t)h + \int_t^{t+h} (t+h-\tau)A''(\tau)\ d\tau \\ &= A(t) + A'(t)h + h^2\int_0^1 (1-x)A''(t+hx)\ dx \qquad \text{where } \tau = t+hx \\ &= A(t) + A'(t)h + B(t,h)h^2 \qquad \text{where } B(t,h) = \int_0^1 (1-x)A''(t+hx)\ dx \end{aligned}$$

Observe that $B(t,h)$ is C^∞ in t and h. Define

$$E(s) = e^{sA(t+h)}e^{(1-s)A(t)}$$

Then

$$\begin{aligned} e^{A(t+h)} - e^{A(t)} &= E(1) - E(0) = \int_0^1 E'(s)\ ds \\ &= \int_0^1 \left\{e^{sA(t+h)}A(t+h)e^{(1-s)A(t)} - e^{sA(t+h)}A(t)e^{(1-s)A(t)}\right\}\ ds \end{aligned}$$

In computing $E'(s)$ we used the product rule and the fact that, for any constant square matrix C, $\frac{d}{ds}e^{sC} = Ce^{sC} = e^{sC}C$. (This is easily proven by expanding the exponentials in power series.) Continuing the computation,

$$\begin{aligned} \tfrac{1}{h}\big[e^{A(t+h)} - e^{A(t)}\big] &= \int_0^1 e^{sA(t+h)}\tfrac{1}{h}\big[A(t+h)-A(t)\big]e^{(1-s)A(t)}\ ds \\ &= \int_0^1 e^{sA(t+h)}\big[A'(t)+B(t,h)h\big]e^{(1-s)A(t)}\ ds \end{aligned}$$

It now suffices to take the limit $h\to 0$. ■