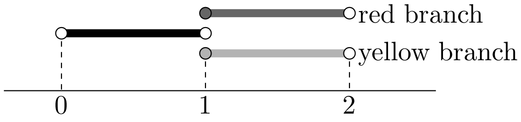
A Degenerate Riemannian Manifold

Define the manifold

$$M = (0,1) \cup \left\{\ (x, \mathrm{red})\ \middle|\ 1 \le x < 2\ \right\} \cup \left\{\ (y, \mathrm{yellow})\ \middle|\ 1 \le y < 2\ \right\}$$

(so that M contains two distinct copies of the interval $[1,2)$ together with one copy of the interval $(0,1)$) with an atlas consisting of the two charts (U,X) and (V,Y) where

$$\begin{array}{lll} U = (0,1) \cup \left\{\ (x, \mathrm{red})\ \middle|\ 1 \le x < 2\ \right\} & X(x) = x \text{ for all } x \in (0,1), & X\big((x,\mathrm{red})\big) = x \text{ for all } x \in [1,2) \\ V = (0,1) \cup \left\{\ (y, \mathrm{yellow})\ \middle|\ 1 \le y < 2\ \right\} & Y(y) = y \text{ for all } y \in (0,1), & Y\big((y,\mathrm{yellow})\big) = y \text{ for all } y \in [1,2) \end{array}$$



X and Y both project vertically downward

These two charts are compatible since

$$U \cap V = (0,1) \qquad X \circ Y^{-1}(y) = y \text{ and } Y \circ X^{-1}(x) = x \text{ for all } x, y \in (0,1)$$

The manifold has dimension one. For each point $m \in U$ we may use $\big\{\frac{\partial}{\partial x}\big|_m\big\}$ as a basis for the tangent space TM_m to M at m. For each point $m \in V$ we may use $\big\{\frac{\partial}{\partial y}\big|_m\big\}$ as a basis for the tangent space TM_m to M at m. In the event that $m \in U \cap V$, then $\frac{\partial}{\partial x}\big|_m = \frac{\partial}{\partial y}\big|_m$. We may turn M into a Riemannian manifold by defining

$$\left\langle \tfrac{\partial}{\partial x}\big|_m, \tfrac{\partial}{\partial x}\big|_m \right\rangle_m = 1 \text{ for all } m \in U \qquad \text{and} \qquad \left\langle \tfrac{\partial}{\partial y}\big|_m, \tfrac{\partial}{\partial y}\big|_m \right\rangle_m = 1 \text{ for all } m \in V$$

I claim that the distance, with respect to this "metric", between the distinct points $(1, \mathrm{red})$ and $(1, \mathrm{yellow})$ is zero. To see this, define for each $0 < \varepsilon < 1$ the curve $q_\varepsilon : [-1,1] \to M$ by

$$q_\varepsilon(t) = \begin{cases} (1, \mathrm{red}) & \text{if } t = -1 \\ 1 - \varepsilon(1 - e^{1+1/t}) & \text{if } -1 < t < 0 \\ 1 - \varepsilon & \text{if } t = 0 \\ 1 - \varepsilon(1 - e^{1-1/t}) & \text{if } 0 < t < 1 \\ (1, \mathrm{yellow}) & \text{if } t = 1 \end{cases}$$

This curve is C^∞ because both local coordinate representations

$$\big(X \circ q_\varepsilon\big)(t) = \begin{cases} 1 - \varepsilon(1 - e^{1+1/t}) & \text{if } -1 \le t < 0 \\ 1 - \varepsilon & \text{if } t = 0 \\ 1 - \varepsilon(1 - e^{1-1/t}) & \text{if } 0 < t < 1 \end{cases} \qquad \big(Y \circ q_\varepsilon\big)(t) = \begin{cases} 1 - \varepsilon(1 - e^{1+1/t}) & \text{if } -1 < t < 0 \\ 1 - \varepsilon & \text{if } t = 0 \\ 1 - \varepsilon(1 - e^{1-1/t}) & \text{if } 0 < t \le 1 \end{cases}$$

are C^∞. The curve q_ε

- starts at $(1, \mathrm{red})$ at time $t = -1$,
- moves to the left into $(0,1)$ coming to a stop at $1 - \varepsilon$ at time $t = 0$ (in fact all derivatives of both $\big(X \circ q_\varepsilon\big)(t)$ and $\big(Y \circ q_\varepsilon\big)(t)$ are zero at $t = 0$)
- then reverses direction and moves to the right in $(0,1)$
- finally ending up at $(1, \mathrm{yellow})$ at time $t = 1$.

The "index shuffling" tangent vectors in the two coordinate patches are

$$\tfrac{d}{dt}\big(X \circ q_\varepsilon\big)(t) = \begin{cases} -\frac{\varepsilon}{t^2}e^{1+1/t} & \text{if } -1 \le t < 0 \\ 0 & \text{if } t = 0 \\ \frac{\varepsilon}{t^2}e^{1-1/t} & \text{if } 0 < t < 1 \end{cases} \qquad \tfrac{d}{dt}\big(Y \circ q_\varepsilon\big)(t) = \begin{cases} -\frac{\varepsilon}{t^2}e^{1+1/t} & \text{if } -1 < t < 0 \\ 0 & \text{if } t = 0 \\ \frac{\varepsilon}{t^2}e^{1-1/t} & \text{if } 0 < t \le 1 \end{cases}$$

So the length of the curve q_ε is

$$\begin{aligned} \big|q_\varepsilon\big| &= \int_{-1}^{1} \sqrt{\langle q_{\varepsilon *t}, q_{\varepsilon *t}\rangle_{q(t)}}\ dt = \int_{-1}^{0} \tfrac{\varepsilon}{t^2}e^{1+1/t}\ dt + \int_{0}^{1} \tfrac{\varepsilon}{t^2}e^{1-1/t}\ dt \overset{s=\frac{1}{t}}{=} \int_{-1}^{-\infty} \varepsilon e^{1+s}\ (-ds) + \int_{\infty}^{1} \varepsilon e^{1-s}\ (-ds) \\ &= \varepsilon\big[-e^{1+s}\big]_{-1}^{-\infty} + \varepsilon\big[e^{1-s}\big]_{\infty}^{1} = 2\varepsilon \end{aligned}$$

By definition, the distance from $(1, \text{red})$ to $(1, \text{yellow})$ is the infimum of the lengths of all curves from $(1, \text{red})$ to $(1, \text{yellow})$. So this distance must be less than or equal to 2ε for all $0 < \varepsilon < 1$. So the distance is exactly zero.

This pathology is generally forbidden by appending to the definition of "manifold" the requirement that it be Hausdorff. This means that if $x, y \in M$ with $x \ne y$, then there exist two open sets $\mathcal{O}$ and $\mathcal{O}'$ in M with $x \in \mathcal{O}$, $y \in \mathcal{O}'$ and $\mathcal{O} \cap \mathcal{O}' = \emptyset$.