Wedge Products of Alternating Forms

Let V be a vector space of dimension $d < \infty$. In these notes, we discuss how to define a wedge product on the space of all alternating forms on $V \times \cdots \times V$'s so as to make it isomorphic to the exterior algebra $\Lambda(V^*)$.

We start by reviewing the equivalence class definition of the exterior algebra over the dual space V^* of V. Define

- $C(V) = \bigoplus_{k=0}^{\infty} V^{*\otimes k}$ (with $V^{*\otimes k}$ being the tensor product of k copies of V^* when $k > 0$ and being $\mathbb{R}$ when $k = 0$) and
- let $I(V^*)$ be the two–sided ideal in $C(V^*)$ generated by $\left\{ \vec{v}^* \otimes \vec{v}^* \ \middle|\ \vec{v}^* \in V^* \right\}$ and
- let, for each $k \geq 2$, $I_k(V^*) = I(V^*) \cap V^{*\otimes k}$ and
- let $\Lambda(V^*) = C(V^*)/I(V^*)$ and
- let $\Lambda^0(V^*) = \mathbb{R}$, $\Lambda^1(V^*) = V^*$ and, for each $2 \leq k \leq d$, $\Lambda^k(V^*) = V^{*\otimes k}/I_k(V^*)$.

Then $\Lambda(V^*)$ is a graded, unital, associative algebra with the operations

$$[c] + [d] = [c + d] \qquad \alpha[c] = [\alpha c] \qquad [c] \wedge [d] = [c \otimes d]$$

for all $c, d \in C(V^*)$ and $\alpha \in \mathbb{R}$. Here $[c]$ is the equivalence class of c under the equivalence relation $c \sim d \iff c - d \in I(V)$. We have that $\Lambda(V^*) = \oplus_{k=0}^{d} \Lambda^k(V^*)$ and that, if $\vec{e}^1, \cdots, \vec{e}^d$ is a basis V^*, then for each $k \geq 2$

$$\left\{ \vec{e}^{i_1} \wedge \cdots \wedge \vec{e}^{i_k} \ \middle|\ 1 \leq i_1 < \cdots < i_k \leq d \right\}$$

is a basis for $\Lambda^k(V^*)$.

Now we start on the multilinear form definition. For each $k \in \mathbb{N}$, denote by $\mathbb{A}^k(V)$ the vector space of all k–linear alternating forms on $V \times \cdots \times V$ (with k factors). That is, $L \in \mathbb{A}$ is a function $L : V \times \cdots \times V \to \mathbb{R}$ (with k factors) that is linear in each of its k arguments and obeys

$$L\big(\vec{v}_1, \cdots, \vec{v}_k\big) = (-1)^{\pi} L\big(\vec{v}_{\pi(1)}, \cdots, \vec{v}_{\pi(k)}\big)$$

for each permutation π of $\{1, \cdots, k\}$. Here $(-1)^{\pi}$ is the sign of the permutation. We denote by S_k the set of all permutations of $\{1, \cdots, k\}$. For L to be nonzero, it is necessary that $k \leq d$.

Of course $\mathbb{A}^1(V)$ is just V^*, as is $\Lambda^1(V^*)$. Both have basis $\{\vec{e}^1, \cdots, \vec{e}^d\}$. To distinguish between $\vec{e}^j$, viewed as an element of $\Lambda^1(V^*)$, and $\vec{e}^j$, viewed as an element of $\mathbb{A}^1(V)$, we rename the latter to E^j. That is, if $\vec{e}_1, \cdots, \vec{e}_d$ is the basis for V that is dual to the basis $\vec{e}^1, \cdots, \vec{e}^d$ for V^*, then $E^j \in \mathbb{A}^1(V)$ is the linear functional

$$E^j\Big(\sum_{k=1}^{d} \alpha^k \vec{e}_k\Big) = \alpha_j$$

So $\{E^1, \cdots, E^d\}$ is a basis for $\mathbb{A}^1(V)$. To construct a basis for $\mathbb{A}^k(V)$, with $2 \le k \le d$, define, for each $1 \le i_1, \cdots, i_k \le d$, the alternating, k–linear form $\varepsilon^{i_1,\cdots,i_k} \in \mathbb{A}^k(V)$ by

$$\varepsilon^{i_1,\cdots,i_k}(\vec{v}_1, \cdots, \vec{v}_k) = \det\big[E^{i_\ell}(\vec{v}_m)\big]_{1\le \ell,m\le k} = \sum_{\pi\in S_k} (-1)^\pi \prod_{\ell=1}^{k} E^{i_\ell}\big(\vec{v}_{\pi(\ell)}\big) \tag{1}$$

If any two i_j's are equal, then $\varepsilon^{i_1,\cdots,i_k} = 0$. If all of the i_j's are distinct, then, for each $\pi \in S_k$, $\varepsilon^{i_1,\cdots,i_k} = (-1)^\pi \varepsilon^{i_{\pi(1)},\cdots,i_{\pi(k)}}$ and, in particular,

$$\varepsilon^{i_1,\cdots,i_k}(\vec{e}_{j_1}, \cdots, \vec{e}_{j_k}) = \begin{cases} (-1)^\pi & \text{if } (j_1, \cdots, j_k) = (i_{\pi(1)}, \cdots, i_{\pi(k)}) \text{ for some permutation } \pi \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\big\{\ \varepsilon^{i_1,\cdots,i_k}\ \big|\ 1 \le i_1 < \cdots < i_k \le d\ \big\}$$

is a basis for $\mathbb{A}^k(V)$.

At this stage $\mathbb{A}(V) = \oplus_{k=0}^d \mathbb{A}^k(V)$ (with $\mathbb{A}^0(V) = \mathbb{R}$) is just a vector space. We would like to endow it with a product $\wedge$ in such a way that $\mathbb{A}(V)$ is isomorphic to $\Lambda(V^*)$. Let us call the isomorphism $\mathcal{I} : \Lambda(V^*) \to \mathbb{A}(V)$. We would like it to preserve the grading. That is, $\mathcal{I}$ should map $\mathbb{A}^k(V)$ onto $\Lambda^k(V^*)$ for each $0 \le k \le d$. It is natural to take the restrictions of $\mathcal{I}$ to $\Lambda^0(V^*) = \mathbb{R} = \mathbb{A}^0(V)$ and to $\Lambda^1(V^*) = V^* = \mathbb{A}^1(V)$ to be identity maps. In particular, to take

$$\mathcal{I}(\vec{e}^{\,k}) = E^k \qquad \text{for all } 1 \le k \le d$$

Once we have turned $\mathbb{A}(V)$ into an algebra by defining $\wedge$, the requirements

- that $\mathcal{I}$ be an algebra isomorphism that preserves the grading
- that $\mathcal{I}(\alpha) = \alpha$ for all $\alpha \in \Lambda^0(V^*) = \mathbb{R}$ and
- that $\mathcal{I}(\vec{e}^{\,k}) = E^k$ for all $1 \le k \le d$

will force

$$\mathcal{I}(\vec{e}^{\,j_1} \wedge \cdots \wedge \vec{e}^{\,j_k}) = E^{j_1} \wedge \cdots \wedge E^{j_k} \in \mathbb{A}^k(V) \qquad \text{for all } 1 \le k \le d,\ 1 \le j_1, \cdots, j_k \le d$$

which completely determines $\mathcal{I}$ by linearity.

Now let us consider how we could define $E^{j_1} \wedge \cdots \wedge E^{j_k} \in \mathbb{A}^k(V)$, for each $2 \le k \le d$. Because $\mathcal{I}$ is to be an isomorphism with $\mathcal{I}(\vec{e}^{\,j_1} \wedge \cdots \wedge \vec{e}^{\,j_k}) = E^{j_1} \wedge \cdots \wedge E^{j_k}$, we must have, for each $2 \le k \le d$,

- $E^{j_{\pi(1)}} \wedge \cdots \wedge E^{j_{\pi(k)}} = (-1)^\pi E^{j_1} \wedge \cdots \wedge E^{j_k} \in \mathbb{A}^k(V)$ for all $\pi \in S_k$ and $1 \le j_1, \cdots, j_k \le d$ and
- $\big\{\ E^{j_1} \wedge \cdots \wedge E^{j_k}\ \big|\ 1 \le j_1 < \cdots < j_k \le d\ \big\}$ must be linearly independent.

These are the only requirements. That is, we may select any collection of alternating forms $\big\{\ E^{j_1,\cdots,j_k}\in\mathbb{A}^k(V)\ \big|\ 2\le k\le d,\ 1\le j_1,\cdots,j_k\le d\ \big\}$ that obey

(H1) $E^{j_{\pi(1)},\cdots,j_{\pi(k)}}=(-1)^\pi E^{j_1,\cdots,j_k}$ for all $\pi\in S_k$ and $1\le j_1,\cdots,j_k\le d$ and

(H2) $\big\{\ E^{j_1,\cdots,j_k}\ \big|\ 1\le j_1<\cdots<j_k\le d\ \big\}$ are linearly independent for each $2\le k\le d$

and then define the wedge product by the distributive rule and

$$E^{i_1,\cdots,i_k}\wedge E^{j_1,\cdots,j_\ell}=E^{i_1,\cdots,i_k,j_1,\cdots,j_\ell} \tag{2}$$

for all $1\le k,\ell\le d$ and $1\le i_1,\cdots,i_k,j_1,\cdots,j_\ell\le d$ (set the right hand side to zero when $k+\ell>d$) and define $\mathcal{I}$ by linearity and

$$\mathcal{I}(\vec{e}^{\,j_1}\wedge\cdots\wedge\vec{e}^{\,j_k})=E^{j_1,\cdots,j_k}\qquad\text{for all } 2\le k\le d \text{ and } 1\le j_1,\cdots,j_k\le d$$

Then $\mathcal{I}$ is an isomorphism and

$$\mathcal{I}(\vec{e}^{\,j_1}\wedge\cdots\wedge\vec{e}^{\,j_k})=E^{j_1,\cdots,j_k}=E^{j_1}\wedge\cdots\wedge E^{j_k}\quad\text{for all } k\ge 1 \text{ and } 1\le j_1,\cdots,j_k\le d \tag{3}$$

Now let's construct a bunch of families of $E^{j_1,\cdots,j_k}$'s satisfying the requirements (H1) and (H2). Each $E^{j_1,\cdots,j_k}$ must, in particular, be a k–linear functional and hence a linear combination

$$E^{j_1,\cdots,j_k}=\sum_{i_1,\cdots,i_k=1}^{d}\alpha^{j_1,\cdots,j_k}_{i_1,\cdots,i_k}\,E^{i_1}\otimes\cdots\otimes E^{i_k}$$

Let's choose our $E^{j_1,\cdots,j_k}$ to be a linear combination of only those $E^{i_1}\otimes\cdots\otimes E^{i_k}$'s with $i_p\in\{j_1,\cdots,j_k\}$ for every $1\le p\le k$.

- Since $\alpha^{j_1,\cdots,j_k}_{i_1,\cdots,i_k}=E^{j_1,\cdots,j_k}\big(\vec{e}_{i_1},\cdots,\vec{e}_{i_k}\big)$ and $E^{j_1,\cdots,j_k}\in\mathbb{A}^k(V)$ is to be alternating, every coefficient $\alpha^{j_1,\cdots,j_k}_{i_1,\cdots,i_k}$ with two or more of the i_p's equal must vanish. Hence the sum over $i_1,\cdots,i_k$ is now restricted to permutations of $j_1,\cdots,j_k$ and

$$E^{j_1,\cdots,j_k}=\sum_{\pi\in S_k}\alpha^{j_1,\cdots,j_k}_{\pi}\,E^{j_{\pi(1)}}\otimes\cdots\otimes E^{j_{\pi(k)}}$$

- Again since

$$\begin{aligned}\alpha^{j_1,\cdots,j_k}_{\pi}&=E^{j_1,\cdots,j_k}\big(\vec{e}_{j_{\pi(1)}},\cdots,\vec{e}_{j_{\pi(k)}}\big)=(-1)^\pi E^{j_1,\cdots,j_k}\big(\vec{e}_{j_1},\cdots,\vec{e}_{j_k}\big)\\&=(-1)^\pi\alpha^{j_1,\cdots,j_k}_{\mathbb{1}}\end{aligned}$$

  where $\mathbb{1}$ is the identity permutation, we now have

$$E^{j_1,\cdots,j_k}=\alpha^{j_1,\cdots,j_k}_{\mathbb{1}}\sum_{\pi\in S_k}(-1)^\pi E^{j_{\pi(1)}}\otimes\cdots\otimes E^{j_{\pi(k)}}=\alpha^{j_1,\cdots,j_k}_{\mathbb{1}}\,\varepsilon^{j_1,\cdots,j_k}$$

To satisfy (H1), it suffices to ensure that $\alpha_{\mathbb{1}}^{j_{\pi(1)},\cdots,j_{\pi(k)}} = \alpha_{\mathbb{1}}^{j_1,\cdots,j_k}$ for all $2 \le k \le d$, $\pi \in S_k$ and $1 \le j_1, \cdots, j_k \le d$. As long as the $\alpha_{\mathbb{1}}^{j_1,\cdots,j_k}$'s are nonzero, (H2) will also be satisfied. To simplify matters further, let's only consider $\alpha_{\mathbb{1}}^{j_1,\cdots,j_k}$'s that depend only on k.

So select, for each $2 \le k \le d$, a nonzero constant a_k and define

$$E^{j_1,\cdots,j_k} = a_k\, \varepsilon^{j_1,\cdots,j_k} \tag{4}$$

for all $1 \le j_1, \cdots, j_k \le d$. Let's determine the resulting wedge product $A \wedge B$ of any $A \in \mathbb{A}^\ell(V)$ and $B \in \mathbb{A}^m(V)$. First consider $A = E^{i_1} \wedge \cdots \wedge E^{i_\ell}$ and $B = E^{j_1} \wedge \cdots \wedge E^{j_m}$. By (2) and (3) and (4),

$$\begin{aligned}
\big((E^{i_1} \wedge \cdots \wedge E^{i_\ell}) \wedge (E^{j_1} \wedge \cdots \wedge E^{j_m})\big)(\vec v_1, \cdots, \vec v_{\ell+m}) &= E^{i_1,\cdots,i_\ell,j_1,\cdots j_m}(\vec v_1, \cdots, \vec v_{\ell+m}) \\
&= a_{\ell+m}\, \varepsilon^{i_1,\cdots,i_\ell,j_1,\cdots j_m}(\vec v_1, \cdots, \vec v_{\ell+m}) \\
&= a_{\ell+m} \sum_{\pi' \in S_{\ell+m}} (-1)^{\pi'} \prod_{p=1}^{\ell} E^{i_p}\big(\vec v_{\pi'(p)}\big) \prod_{p=1}^{m} E^{j_{\ell+p}}\big(\vec v_{\pi'(\ell+p)}\big) \qquad \text{by (1)} \\
&= \tfrac{a_{\ell+m}}{\ell!\, m!} \sum_{\sigma \in S_\ell} \sum_{\tau \in S_m} \sum_{\pi' \in S_{\ell+m}} (-1)^{\pi'} \prod_{p=1}^{\ell} E^{i_p}\big(\vec v_{\pi'(p)}\big) \prod_{p=1}^{m} E^{j_{\ell+p}}\big(\vec v_{\pi'(\ell+p)}\big) \\
&= \tfrac{a_{\ell+m}}{\ell!\, m!} \sum_{\sigma \in S_\ell} \sum_{\tau \in S_m} \sum_{\pi \in S_{\ell+m}} (-1)^{\pi}(-1)^{\sigma}(-1)^{\tau} \prod_{p=1}^{\ell} E^{i_p}\big(\vec v_{\pi(\sigma(p))}\big) \prod_{p=1}^{m} E^{j_{\ell+p}}\big(\vec v_{\pi(\ell+\tau(p))}\big) \\
&= \tfrac{a_{\ell+m}}{\ell!\, m!} \sum_{\pi \in S_{\ell+m}} (-1)^{\pi} \varepsilon^{i_1,\cdots,i_\ell}(\vec v_{\pi(1)}, \cdots, \vec v_{\pi(\ell)})\ \varepsilon^{j_1,\cdots j_m}\big(\vec v_{\pi(\ell+1)}, \cdots, \vec v_{\pi(\ell+m)}\big) \\
&= \tfrac{a_{\ell+m}}{a_\ell a_m} \tfrac{1}{\ell!\, m!} \sum_{\pi \in S_{\ell+m}} (-1)^{\pi} E^{i_1,\cdots,i_\ell}(\vec v_{\pi(1)}, \cdots, \vec v_{\pi(\ell)})\ E^{j_1,\cdots j_m}\big(\vec v_{\pi(\ell+1)}, \cdots, \vec v_{\pi(\ell+m)}\big) \\
&= \tfrac{a_{\ell+m}}{a_\ell a_m} \tfrac{1}{\ell!\, m!} \sum_{\pi \in S_{\ell+m}} (-1)^{\pi} (E^{i_1} \wedge \cdots \wedge E^{i_\ell})(\vec v_{\pi(1)}, \cdots, \vec v_{\pi(\ell)}) \\
&\qquad\qquad (E^{j_1} \wedge \cdots \wedge E^{j_m})(\vec v_{\pi(\ell+1)}, \cdots, \vec v_{\pi(\ell+m)}\big)
\end{aligned}$$

In going from the fourth line to the fifth line we made the change of summation variable (in the π' sum, for each fixed σ and τ) $\pi' = \pi \circ (\sigma, \tau)$, where $(\sigma, \tau) \in S_{k+\ell}$ acts by $p \to \sigma(p)$ for $1 \le p \le \ell$ and $\ell + p \to \ell + \tau(p)$ for $1 \le p \le m$. By distributivity, under the wedge product determined by (2) and (4), we have, for all $A \in \mathbb{A}^\ell(V)$ and $B \in \mathbb{A}^m(V)$

$$\begin{aligned}
(A \wedge B)(\vec v_1, \cdots, \vec v_{\ell+m}) &= \tfrac{a_{\ell+m}}{a_\ell a_m} \tfrac{1}{\ell!} \tfrac{1}{m!} \sum_{\pi \in S_{\ell+m}} (-1)^{\pi} A(\vec v_{\pi(1)}, \cdots, \vec v_{\pi(\ell)})\ B(\vec v_{\pi(\ell+1)}, \cdots, \vec v_{\pi(\ell+m)}) \\
&= \tfrac{a_{\ell+m}}{a_\ell a_m} \sum_{(I,J) \in \Pi_{\ell,m}(\ell+m)} \operatorname{sgn}(I, J)\ A(\vec v_I)\ B(\vec v_J)
\end{aligned}$$

where $\Pi_{\ell,m}(\ell+m)$ is the set of all partitions of $\{1, \cdots, \ell+m\}$ into two disjoint subsets $I = \{i_1 < \cdots < i_\ell\}$ and $J = \{j_1 < \cdots < j_m\}$ of size ℓ and m, respectively, and $\operatorname{sgn}(I, J)$

is the sign of the permutation that reorders $(i_1, \cdots i_\ell, j_1, \cdots, j_m)$ to $(1, \cdots, \ell + m)$. The shorthand notations $\vec{v}_I = \vec{v}_{i_1}, \cdots, \vec{v}_{i_\ell}$ and $\vec{v}_J = \vec{v}_{j_1}, \cdots, \vec{v}_{j_m}$.

There are two choices of the a_k's that are in common use.

(C1) The first, which we shall use, is $a_k = 1$. For this choice (which is used in the introductory differential geometry texts by Warner, Lee, do Carmo and Spivak),

$$\begin{aligned}(A \wedge B)(\vec{v}_1, \cdots, \vec{v}_{\ell+m}) &= \tfrac{1}{\ell!}\tfrac{1}{m!} \sum_{\pi \in S_{\ell+m}} (-1)^\pi A(\vec{v}_{\pi(1)}, \cdots, \vec{v}_{\pi(\ell)})\ B(\vec{v}_{\pi(\ell+1)}, \cdots, \vec{v}_{\pi(\ell+m)}) \\ &= \sum_{(I,J) \in \Pi_{\ell,m}(\ell+m)} \operatorname{sgn}(I, J)\ A(\vec{v}_I)\ B(\vec{v}_J)\end{aligned}$$

and

$$\begin{aligned}(E^{i_1} \wedge \cdots \wedge E^{i_k})(\vec{v}_1, \cdots, \vec{v}_k) = \varepsilon^{i_1,\cdots,i_k}(\vec{v}_1, \cdots, \vec{v}_k) &= \sum_{\pi \in S_k} (-1)^\pi \prod_{\ell=1}^{k} E^{i_\ell}\big(\vec{v}_{\pi(\ell)}\big) \\ &= \det\big[E^{i_\ell}(\vec{v}_m)\big]_{1 \le \ell, m \le k}\end{aligned}$$

This choice has the advantage that, in $\mathbb{R}^d$ with the standard basis,

$$(E^1 \wedge \cdots \wedge E^d)(\vec{v}_1, \cdots, \vec{v}_d) = \det\big[v_m^\ell\big]_{1 \le \ell, m \le k}$$

(where v_m^ℓ is the $\ell^{\rm th}$ component of $\vec{v}_m$) is just the usual volume of the parallelepiped with edges $\vec{v}_1, \cdots, \vec{v}_d$.

(C2) The second is $a_k = \frac{1}{k!}$. For this choice (which is used in the text by Kobayashi and Nomizu),

$$(A \wedge B)(\vec{v}_1, \cdots, \vec{v}_{\ell+m}) = \tfrac{1}{(\ell+m)!} \sum_{\pi \in S_{\ell+m}} (-1)^\pi A(\vec{v}_{\pi(1)}, \cdots, \vec{v}_{\pi(\ell)})\ B(\vec{v}_{\pi(\ell+1)}, \cdots, \vec{v}_{\pi(\ell+m)})$$

and

$$(E^{i_1} \wedge \cdots \wedge E^{i_k})(\vec{v}_1, \cdots, \vec{v}_k) = \tfrac{1}{(\ell+m)!} \det\big[E^{i_\ell}(\vec{v}_m)\big]_{1 \le \ell, m \le k}$$

This choice has the advantage that $E^{i_1} \wedge \cdots \wedge E^{i_k}$ is the unique multilinear form in the equivalence class $\big[e^{i_1} \otimes \cdots \otimes e^{i_k}\big]$ that is alternating.