Wedge Products of Alternating Forms

Let $V$ be a vector space of dimension $d < \infty$. In these notes, we discuss how to define a wedge product on the space of all alternating forms on $V \times \cdots \times V$'s so as to make it isomorphic to the exterior algebra $\Lambda(V^*)$.

We start by reviewing the equivalence class definition of the exterior algebra over the dual space $V^*$ of $V$. Define

- $C(V) = \bigoplus_{k=0}^{\infty} V^* \otimes^k$ (with $V^* \otimes^k$ being the tensor product of $k$ copies of $V^*$ when $k > 0$ and being $\mathbb{R}$ when $k = 0$) and
- let $I(V^*)$ be the two-sided ideal in $C(V^*)$ generated by $\{ \bar{v}^* \otimes \bar{v}^* \mid \bar{v}^* \in V^* \}$ and
- let, for each $k \geq 2$, $I_k(V^*) = I(V^*) \cap V^* \otimes^k$ and
- let $\Lambda(V^*) = C(V^*)/I(V^*)$ and
- let $\Lambda^0(V^*) = \mathbb{R}$, $\Lambda^1(V^*) = V^*$ and, for each $2 \leq k \leq d$, $\Lambda^k(V^*) = V^* \otimes^k/I_k(V^*)$.

Then $\Lambda(V^*)$ is a graded, unital, associative algebra with the operations

$$[c] + [d] = [c + d] \quad \alpha[c] = [\alpha c] \quad [c] \wedge [d] = [c \otimes d]$$

for all $c, d \in C(V^*)$ and $\alpha \in \mathbb{R}$. Here $[c]$ is the equivalence class of $c$ under the equivalence relation $c \sim d \iff c - d \in I(V)$. We have that $\Lambda(V^*) = \bigoplus_{k=0}^d \Lambda^k(V^*)$ and that, if $\bar{e}^1, \ldots, \bar{e}^d$ is a basis $V^*$, then for each $k \geq 2$

$$\{ \bar{e}^1 \wedge \cdots \wedge \bar{e}^k \mid 1 \leq i_1 < \cdots < i_k \leq d \}$$

is a basis for $\Lambda^k(V^*)$.

Now we start on the multilinear form definition. For each $k \in \mathbb{N}$, denote by $A^k(V)$ the vector space of all $k$–linear alternating forms on $V \times \cdots \times V$ (with $k$ factors). That is, $L \in A$ is a function $L : V \times \cdots \times V \to \mathbb{R}$ (with $k$ factors) that is linear in each of its $k$ arguments and obeys

$$L(\bar{v}_1, \cdots, \bar{v}_k) = (-1)^\pi L(\bar{v}_{\pi(1)}, \cdots, \bar{v}_{\pi(k)})$$

for each permutation $\pi$ of $\{1, \cdots, k\}$. Here $(-1)^\pi$ is the sign of the permutation. We denote by $S_k$ the set of all permutations of $\{1, \cdots, k\}$. For $L$ to be nonzero, it is necessary that $k \leq d$.

Of course $A^1(V)$ is just $V^*$, as is $A^1(V^*)$. Both have basis $\{\bar{e}^1, \cdots, \bar{e}^d\}$. To distinguish between $\bar{e}^i$, viewed as an element of $A^1(V^*)$, and $\bar{e}^i$, viewed as an element of $A^1(V)$, we rename the latter to $E^j$. That is, if $\bar{e}_1, \cdots, \bar{e}_d$ is the basis for $V$ that is dual to the basis $\bar{e}^1, \cdots, \bar{e}^d$ for $V^*$, then $E^j \in A^1(V)$ is the linear functional

$$E^j \left( \sum_{k=1}^d \alpha^k \bar{e}_k \right) = \alpha_j$$

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So \( \{E^1, \cdots, E^d\} \) is a basis for \( \Lambda^1(V) \). To construct a basis for \( \Lambda^k(V) \), with \( 2 \leq k \leq d \), define, for each \( 1 \leq i_1, \cdots, i_k \leq d \), the alternating, \( k \)-linear form \( \varepsilon^{i_1, \cdots, i_k} \in \Lambda^k(V) \) by

\[
\varepsilon^{i_1, \cdots, i_k} (\vec{v}_1, \cdots, \vec{v}_k) = \det \left[ E^{i_\ell} (\vec{v}_m) \right]_{1 \leq \ell, m \leq k} = \sum_{\pi \in S_k} (-1)^\pi \prod_{\ell=1}^k E^{i_\ell} (\vec{v}_{\pi(\ell)}) \quad (1)
\]

If any two \( i_j \)'s are equal, then \( \varepsilon^{i_1, \cdots, i_k} = 0 \). If all of the \( i_j \)'s are distinct, then, for each \( \pi \in S_k \), \( \varepsilon^{i_1, \cdots, i_k} = (-1)^\pi \varepsilon^{i_{\pi(1)}, \cdots, i_{\pi(k)}} \) and, in particular,

\[
\varepsilon^{i_1, \cdots, i_k} (\vec{e}_{j_1}, \cdots, \vec{e}_{j_k}) = \begin{cases} (-1)^\pi & \text{if } (j_1, \cdots, j_k) = (i_{\pi(1)}, \cdots, i_{\pi(k)}) \text{ for some permutation } \pi \\ 0 & \text{otherwise} \end{cases}
\]

Then

\[
\{ \varepsilon^{i_1, \cdots, i_k} \mid 1 \leq i_1 < \cdots < i_k \leq d \}
\]

is a basis for \( \Lambda^k(V) \).

At this stage \( \Lambda(V) = \bigoplus_{k=0}^d \Lambda^k(V) \) (with \( \Lambda^0(V) = \mathbb{R} \)) is just a vector space. We would like to endow it with a product \( \wedge \) in such a way that \( \Lambda(V) \) is isomorphic to \( \Lambda(V^*) \). Let us call the isomorphism \( \mathcal{I} : \Lambda(V^*) \rightarrow \Lambda(V) \). We would like it to preserve the grading. That is, \( \mathcal{I} \) should map \( \Lambda^k(V) \) onto \( \Lambda^k(V^*) \) for each \( 0 \leq k \leq d \). It is natural to take the restrictions of \( \mathcal{I} \) to \( \Lambda^0(V^*) = \mathbb{R} = \Lambda^0(V) \) and to \( \Lambda^1(V^*) = V^* = \Lambda^1(V) \) to be identity maps. In particular, to take

\[
\mathcal{I}(e^k) = E^k \quad \text{for all } 1 \leq k \leq d
\]

Once we have turned \( \Lambda(V) \) into an algebra by defining \( \wedge \), the requirements

\begin{itemize}
\item \( \mathcal{I} \) be an algebra isomorphism that preserves the grading
\item \( \mathcal{I}(\alpha) = \alpha \) for all \( \alpha \in \Lambda^0(V^*) = \mathbb{R} \) and
\item \( \mathcal{I}(e^k) = E^k \) for all \( 1 \leq k \leq d \)
\end{itemize}

will force

\[
\mathcal{I}(\vec{e}^{j_1} \wedge \cdots \wedge \vec{e}^{j_k}) = E^{j_1} \wedge \cdots \wedge E^{j_k} \in \Lambda^k(V) \quad \text{for all } 1 \leq k \leq d, \ 1 \leq j_1, \cdots, j_k \leq d
\]

which completely determines \( \mathcal{I} \) by linearity.

Now let us consider how we could define \( E^{j_1} \wedge \cdots \wedge E^{j_k} \in \Lambda^k(V) \), for each \( 2 \leq k \leq d \). Because \( \mathcal{I} \) is to be an isomorphism with \( \mathcal{I}(\vec{e}^{j_1} \wedge \cdots \wedge \vec{e}^{j_k}) = E^{j_1} \wedge \cdots \wedge E^{j_k} \), we must have, for each \( 2 \leq k \leq d \),

\begin{itemize}
\item \( E^{j_{\pi(1)}} \wedge \cdots \wedge E^{j_{\pi(k)}} = (-1)^\pi E^{j_1} \wedge \cdots \wedge E^{j_k} \in \Lambda^k(V) \) for all \( \pi \in S_k \) and \( 1 \leq j_1, \cdots, j_k \leq d \)
\item \( \{ E^{j_1} \wedge \cdots \wedge E^{j_k} \mid 1 \leq j_1 < \cdots < j_k \leq d \} \) must be linearly independent.
\end{itemize}
These are the only requirements. That is, we may select any collection of alternating forms 
\[ \{ E^{j_1, \ldots, j_k} \in \mathbb{A}^k(V) \mid 2 \leq k \leq d, \ 1 \leq j_1, \ldots, j_k \leq d \} \]
that obey

(H1) \[ E^{j_1, \ldots, j_k} = (-1)^\pi E^{j_{\pi(1)}, \ldots, j_{\pi(k)}} \] \text{ for all } \pi \in S_k \text{ and } 1 \leq j_1, \ldots, j_k \leq d \]

(H2) \[ \{ E^{j_1, \ldots, j_k} \mid 1 \leq j_1 < \ldots < j_k \leq d \} \] \text{ are linearly independent for each } 2 \leq k \leq d

and then define the wedge product by the distributive rule and

\[ E^{i_1, \ldots, i_k} \wedge E^{j_1, \ldots, j_\ell} = E^{i_1, \ldots, i_k, j_1, \ldots, j_\ell} \] \hspace{1cm} (2)

for all \( 1 \leq k, \ell \leq d \) and \( 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_\ell \leq d \) (set the right hand side to zero when \( k + \ell > d \)) and define \( \mathcal{I} \) by linearity and

\[ \mathcal{I}(\bar{e}^{j_1} \wedge \ldots \wedge \bar{e}^{j_k}) = E^{j_1, \ldots, j_k} \] \text{ for all } 2 \leq k \leq d \text{ and } 1 \leq j_1, \ldots, j_k \leq d

Then \( \mathcal{I} \) is an isomorphism and

\[ \mathcal{I}(\bar{e}^{j_1} \wedge \ldots \wedge \bar{e}^{j_k}) = E^{j_1, \ldots, j_k} = E^{j_1} \wedge \ldots \wedge E^{j_k} \] \text{ for all } k \geq 1 \text{ and } 1 \leq j_1, \ldots, j_k \leq d \hspace{1cm} (3)

Now let’s construct a bunch of families of \( E^{j_1, \ldots, j_k} \)'s satisfying the requirements (H1) and (H2). Each \( E^{j_1, \ldots, j_k} \) must, in particular, be a \( k \)-linear functional and hence a linear combination

\[ E^{j_1, \ldots, j_k} = \sum_{i_1, \ldots, i_k=1}^{d} \alpha^{j_1, \ldots, j_k}_{i_1, \ldots, i_k} E^{i_1} \otimes \ldots \otimes E^{i_k} \]

Let’s choose our \( E^{j_1, \ldots, j_k} \) to be a linear combination of only those \( E^{i_1} \otimes \ldots \otimes E^{i_k} \)'s with \( i_p \in \{ j_1, \ldots, j_k \} \) for every \( 1 \leq p \leq k \).

\[ \circ \] Since \( \alpha^{j_1, \ldots, j_k}_{i_1, \ldots, i_k} = E^{j_1, \ldots, j_k}(\bar{e}_{i_1}, \ldots, \bar{e}_{i_k}) \) and \( E^{j_1, \ldots, j_k} \in \mathbb{A}^k(V) \) is to be alternating, every coefficient \( \alpha^{j_1, \ldots, j_k}_{i_1, \ldots, i_k} \) with two or more of the \( i_p \)'s equal must vanish. Hence the sum over \( i_1, \ldots, i_k \) is now restricted to permutations of \( j_1, \ldots, j_k \) and

\[ E^{j_1, \ldots, j_k} = \sum_{\pi \in S_k} \alpha^{j_1, \ldots, j_k}_{\pi} E^{j_{\pi(1)}} \otimes \ldots \otimes E^{j_{\pi(k)}} \]

\[ \circ \] Again since

\[ \alpha^{j_1, \ldots, j_k}_{\pi} = E^{j_1, \ldots, j_k}(\bar{e}_{j_{\pi(1)}}, \ldots, \bar{e}_{j_{\pi(k)}}) = (-1)^\pi E^{j_1, \ldots, j_k}(\bar{e}_{j_1}, \ldots, \bar{e}_{j_k}) \]

\[ = (-1)^\pi \alpha^{j_1, \ldots, j_k}_{\mathbb{1}} \]

where \( \mathbb{1} \) is the identity permutation, we now have

\[ E^{j_1, \ldots, j_k} = \alpha^{j_1, \ldots, j_k}_{\mathbb{1}} \sum_{\pi \in S_k} (-1)^\pi E^{j_{\pi(1)}} \otimes \ldots \otimes E^{j_{\pi(k)}} = \alpha^{j_1, \ldots, j_k}_{\mathbb{1}} \cdot \bar{e}_{j_1} \wedge \ldots \wedge \bar{e}_{j_k} \]
To satisfy (H1), it suffices to ensure that \( \alpha^j_{\pi(1), \cdots, \pi(k)} = \alpha^{j_1, \cdots, j_k} \) for all \( 2 \leq k \leq d, \pi \in S_k \) and \( 1 \leq j_1, \cdots, j_k \leq d \). As long as the \( \alpha^{j_1, \cdots, j_k} \)'s are nonzero, (H2) will also be satisfied. To simplify matters further, let's only consider \( \alpha^{j_1, \cdots, j_k} \)'s that depend only on \( k \).

So select, for each \( 2 \leq k \leq d \), a nonzero constant \( a_k \) and define

\[
E^{j_1, \cdots, j_k} = a_k \epsilon^{j_1, \cdots, j_k}
\]

for all \( 1 \leq j_1, \cdots, j_k \leq d \). Let's determine the resulting wedge product \( A \wedge B \) of any \( A \in \mathbb{A}^\ell(V) \) and \( B \in \mathbb{A}^m(V) \). First consider \( A = E^{i_1} \wedge \cdots \wedge E^{i_\ell} \) and \( B = E^{j_1} \wedge \cdots \wedge E^{j_m} \). By (2) and (3) and (4),

\[
((E^{i_1} \wedge \cdots \wedge E^{i_\ell}) \wedge (E^{j_1} \wedge \cdots \wedge E^{j_m}))(\vec{v}_1, \cdots, \vec{v}_{\ell+m}) = E^{i_1, \cdots, i_\ell, j_1, \cdots, j_m}(\vec{v}_1, \cdots, \vec{v}_{\ell+m})
\]

\[
= a_{\ell+m} \prod_{p=1}^{\ell} E^{i_p}(\vec{v}_{\pi'(p)}) \prod_{p=1}^{m} E^{j_{\ell+p}}(\vec{v}_{\pi'(\ell+p)}) \quad \text{by (1)}
\]

\[
= a_{\ell+m} \left[ \sum_{\pi' \in S_{\ell+m}} (-1)^{\pi'} \prod_{p=1}^{\ell} E^{i_p}(\vec{v}_{\pi'(p)}) \prod_{p=1}^{m} E^{j_{\ell+p}}(\vec{v}_{\pi'(\ell+p)}) \right]
\]

\[
= \frac{a_{\ell+m}}{\ell! m!} \sum_{\sigma \in S_\ell} \sum_{\pi' \in S_{\ell+m}} \sum_{\pi' \in S_{\ell+m}} (-1)^{\pi'} (-1)^\sigma (-1) \prod_{p=1}^{\ell} E^{i_p}(\vec{v}_{\pi'(p)}) \prod_{p=1}^{m} E^{j_{\ell+p}}(\vec{v}_{\pi'(\ell+p)})
\]

\[
= \frac{a_{\ell+m}}{\ell! m!} \sum_{\pi' \in S_{\ell+m}} (-1)^{\pi'} E^{i_1, \cdots, i_\ell}(\vec{v}_{\pi'(1)}, \cdots, \vec{v}_{\pi'(\ell)}) E^{j_1, \cdots, j_m}(\vec{v}_{\pi'(\ell+1)}, \cdots, \vec{v}_{\pi'(\ell+m)})
\]

\[
= \frac{a_{\ell+m}}{\ell! m!} \sum_{\pi \in S_{\ell+m}} (-1)^{\pi} (E^{i_1} \wedge \cdots \wedge E^{i_\ell})(\vec{v}_{\pi(1)}, \cdots, \vec{v}_{\pi(\ell)}) E^{j_1, \cdots, j_m}(\vec{v}_{\pi(\ell+1)}, \cdots, \vec{v}_{\pi(\ell+m)})
\]

In going from the fourth line to the fifth line we made the change of summation variable (in the \( \pi' \) sum, for each fixed \( \sigma \) and \( \tau \) \( \pi' = \pi \circ (\sigma, \tau) \), where \( (\sigma, \tau) \in S_{k+\ell} \) acts by \( p \to \sigma(p) \) for \( 1 \leq p \leq \ell \) and \( \ell + p \to \ell + \tau(p) \) for \( 1 \leq p \leq m \). By distributivity, under the wedge product determined by (2) and (4), we have, for all \( A \in \mathbb{A}^\ell(V) \) and \( B \in \mathbb{A}^m(V) \)

\[
(A \wedge B)(\vec{v}_1, \cdots, \vec{v}_{\ell+m}) = \frac{a_{\ell+m}}{\ell! m!} \sum_{\pi \in S_{\ell+m}} (-1)^\pi A(\vec{v}_{\pi(1)}, \cdots, \vec{v}_{\pi(\ell)}) B(\vec{v}_{\pi(\ell+1)}, \cdots, \vec{v}_{\pi(\ell+m)})
\]

\[
= \frac{a_{\ell+m}}{\ell! m!} \sum_{(I, J) \in \Pi_{\ell, m}(\ell+m)} \text{sgn}(I, J) A(\vec{v}_I) B(\vec{v}_J)
\]

where \( \Pi_{\ell, m}(\ell + m) \) is the set of all partitions of \( \{1, \cdots, \ell + m\} \) into two disjoint subsets \( I = \{i_1 < \cdots < i_\ell\} \) and \( J = \{j_1 < \cdots < j_m\} \) of size \( \ell \) and \( m \), respectively, and \( \text{sgn}(I, J) \)
is the sign of the permutation that reorders \((i_1, \ldots, i_\ell, j_1, \ldots, j_m)\) to \((1, \ldots, \ell + m)\). The
shorthand notations \(\vec{v}_I = \vec{v}_{i_1}, \ldots, \vec{v}_{i_\ell}\) and \(\vec{v}_J = \vec{v}_{j_1}, \ldots, \vec{v}_{j_m}\).

There are two choices of the \(a_k\)'s that are in common use.

(C1) The first, which we shall use, is \(a_k = 1\). For this choice (which is used in the
introductory differential geometry texts by Warner, Lee, do Carmo and Spivak),

\[
(A \wedge B)(\vec{v}_1, \ldots, \vec{v}_{\ell+m}) = \frac{1}{\ell! m!} \sum_{\pi \in S_{\ell+m}} (-1)^\pi A(\vec{v}_{\pi(1)}, \ldots, \vec{v}_{\pi(\ell)}) \cdot B(\vec{v}_{\pi(\ell+1)}, \ldots, \vec{v}_{\pi(\ell+m)})
\]

\[
= \sum_{(I,J) \in \Pi_{\ell,m}(\ell+m)} \text{sgn}(I, J) A(\vec{v}_I) \cdot B(\vec{v}_J)
\]

and

\[
(E^{i_1} \wedge \cdots \wedge E^{i_k})(\vec{v}_1, \ldots, \vec{v}_k) = \varepsilon^{i_1, \cdots, i_k}(\vec{v}_1, \ldots, \vec{v}_k) = \sum_{\pi \in S_k} (-1)^\pi \prod_{\ell=1}^k E^{i_\ell}(\vec{v}_{\pi(\ell)})
\]

\[
= \det [E^{i_\ell}(\vec{v}_m)]_{1 \leq \ell, m \leq k}
\]

This choice has the advantage that, in \(\mathbb{R}^d\) with the standard basis,

\[
(E^1 \wedge \cdots \wedge E^d)(\vec{v}_1, \ldots, \vec{v}_d) = \det [v^\ell_m]_{1 \leq \ell, m \leq k}
\]

(where \(v^\ell_m\) is the \(\ell\)th component of \(\vec{v}_m\)) is just the usual volume of the parallelepiped with
edges \(\vec{v}_1, \ldots, \vec{v}_d\).

(C2) The second is \(a_k = \frac{1}{k!}\). For this choice (which is used in the text by Kobayashi and
Nomizu),

\[
(A \wedge B)(\vec{v}_1, \ldots, \vec{v}_{\ell+m}) = \frac{1}{(\ell+m)!} \sum_{\pi \in S_{\ell+m}} (-1)^\pi A(\vec{v}_{\pi(1)}, \ldots, \vec{v}_{\pi(\ell)}) \cdot B(\vec{v}_{\pi(\ell+1)}, \ldots, \vec{v}_{\pi(\ell+m)})
\]

and

\[
(E^{i_1} \wedge \cdots \wedge E^{i_k})(\vec{v}_1, \ldots, \vec{v}_k) = \frac{1}{(\ell+m)!} \det [E^{i_\ell}(\vec{v}_m)]_{1 \leq \ell, m \leq k}
\]

This choice has the advantage that \(E^{i_1} \wedge \cdots \wedge E^{i_k}\) is the unique multilinear form in the
equivalence class \([e^{i_1} \otimes \cdots \otimes e^{i_k}]\) that is alternating.