Resumé of definitions of "tangent vector"

Let $\mathcal{M}$ be a manifold, $m \in \mathcal{M}$, and $\mathbf{X} = (X^1, \cdots X^d) : \mathcal{U} \to \mathbb{R}^d$ be coordinates in an open neighbourhood $\mathcal{U}$ of m. Le $q : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathcal{M}$ be a C^∞ curve with $q(t_0) = m$.

Index shuffling definition

$$\text{tangent vector to } q(t) \text{ at } m \text{ in } \mathbf{X}\text{–coordinates} = \tfrac{d}{dt}\mathbf{X}\big(q(t)\big)\big|_{t=t_0}$$

A tangent vector at m is an equivalence class of pairs $\big(\mathbf{U}, (\mathcal{U}, \mathbf{X})\big)$, where $\mathbf{U} \in \mathbb{R}^d$ and $(\mathcal{U}, \mathbf{X})$ is a coordinate system with $m \in \mathcal{U}$, under the equivalence relation that $\big(\mathbf{U}, (\mathcal{U}, \mathbf{X})\big) \sim \big(\mathbf{V}, (\mathcal{V}, \mathbf{Y})\big)$ if $\mathbf{U}$ and $\mathbf{V}$ obey $(*)$, below.

Equivalence class of curves definition

$$\text{tangent vector to } q(t) \text{ at } m = [q] = \Big\{\ p\ \Big|\ p(t_1) = m,\ \tfrac{d}{dt}\mathbf{X}\big(q(t)\big)\big|_{t=t_0} = \tfrac{d}{dt}\mathbf{X}\big(p(t)\big)\big|_{t=t_1}\ \Big\}$$

Directional derivative definition

Let C_m^∞ be the set of real–valued functions that are defined and C^∞ on a neighbourhood of m. A tangent vector at m is a map

$$D : C_m^\infty \to \mathbb{R}$$

obeying

$$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$$
$$D(fg) = f(m)D(g) + g(m)D(f)$$

for all $\alpha, \beta \in \mathbb{R},\ f, g \in C_m^\infty$. Such a D is called a derivation at m.

Change of coordinates

Let $\mathbf{X} = (X^1, \cdots X^d)$ and $\mathbf{Y} = (Y^1, \cdots Y^d)$ be two coordinate systems near m and let $\mathbf{U} = \frac{d}{dt}\mathbf{X}\big(q(t)\big)\big|_{t=0}$ and $\mathbf{V} = \frac{d}{dt}\mathbf{Y}\big(q(t)\big)\big|_{t=0}$ be the tangent vectors in $\mathbf{X}$– and $\mathbf{Y}$–coordinates respectively. Then

$$V^i = \sum_{j=1}^{d} \tfrac{\partial\, Y^i \circ \mathbf{X}^{-1}}{\partial x^j}\big(\mathbf{X}(m)\big)U^j \qquad (*)$$

Relationships

If $q : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathcal{M}$ is a C^∞ curve with $q(t_0) = m$, then

$$q_{*t_0}(f) = \tfrac{d}{dt} f\big(q(t)\big)\big|_{t=t_0}$$

defines a derivation at m. The map $[q] \mapsto q_{*t_0}$ is a well-defined 1–1, onto map from the set of all equivalence classes of curves to the set of all derivations.

Let $\mathbf{X} = (X^1, \cdots X^d)$ be a coordinate system near m. Then $\left\{\frac{\partial}{\partial x^i}\big|_m\right\}_{1\le i\le d}$ is a basis for the tangent space $\mathcal{M}_m$ to $\mathcal{M}$ at m, where $\frac{\partial}{\partial x^i}\big|_m$ is the derivation at m defined by

$$\tfrac{\partial}{\partial x^i}\big|_m f = \tfrac{\partial}{\partial x^i} f \circ \mathbf{X}^{-1}(\vec{x})\big|_{\vec{x}=\mathbf{X}(m)}$$

That is, each derivation at m has a unique representation of the form

$$D = \sum_{i=1}^{d} U^i \tfrac{\partial}{\partial x^i}\big|_m$$

If $D = q_{*t_0}$, then $\mathbf{U} = \frac{d}{dt}\mathbf{X}\big(q(t)\big)\big|_{t=t_0}$ is the tangent vector to $q(t)$ at m in $\mathbf{X}$–coordinates.