A Little Point Set Topology

A topological space is a generalization of a metric space that allows one to talk about limits, convergence, continuity and so on without requiring the concept of a distance between points in the space. It is based on the observation that, in the theory of metric spaces, the definitions of limits and continuity can be formulated without the word "metric" appearing in the definitions, if you know what the open subsets of the metric space are. Here is the definition.

Definition 1 (Topological Space) A topological space $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets of X that satisfies the axioms

- ▷ $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
- ▷ The union of any collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.
- ▷ The intersection of any finite collection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

The sets in $\mathcal{T}$ are called the open subsets of X. Usually when you are talking about a topological space $(X, \mathcal{T})$, it is clear from the context what $\mathcal{T}$ is. Then you usually just refer to the "topological space X". Once you have specified what sets are open, you can import a large number of definitions from the theory of metric spaces.

Definition 2 Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces.

(a) A subset $C \subset X$ is said to be closed if its complement $X \setminus C \in \mathcal{T}$. Of course
  - ▷ $\emptyset$ and X are closed
  - ▷ The intersection of any collection of closed sets is also closed.
  - ▷ The union of any finite collection of closed sets is also closed.

(b) The closure $\bar{A}$ of a subset $A \subset X$ is the intersection of all closed subsets that contain A. Of course, $\bar{A}$ is closed and is the smallest closed subset containing A.

(c) The interior A° of a subset $A \subset X$ is the union of all open subsets that are contained in A. Of course, A° is open and is the largest open subset contained in A.

(d) We'll define a neighbourhood of a point $x \in X$ to be an open set N with $x \in N$. Some people define a neighbourhood of a point $x \in X$ to be a set N with $x \in N^\circ$.

(e) A sequence of points $x_1, x_2, x_3, \cdots$ in X is said to converge to $x \in X$, if for each $\mathcal{O} \in \mathcal{T}$ with $x \in \mathcal{O}$, there is a natural number N such that $x_n \in \mathcal{O}$ for all $n \geq N$.

(f) A subset $C \subset X$ is said to be compact if every open cover of C has a finite subcover. That is, if $\{\mathcal{O}_j\}_{j \in \mathcal{J}}$ is a collection of open subsets of X with $C \subset \bigcup_{j \in \mathcal{J}} \mathcal{O}_j$, then there is a finite set $\mathcal{I} \subset \mathcal{J}$ of indices such that $C \subset \bigcup_{j \in \mathcal{I}} \mathcal{O}_j$ too.

(g) A subset $C \subset X$ is said to be countably compact if every countable open cover of C has a finite subcover.

(h) A subset $C \subset X$ is said to be sequentially compact if every sequence of points in C has a subsequence that converges to a point of C.

(i) A subset $C \subset X$ is said to be disconnected if there are two disjoint sets $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$ with $\mathcal{O}_1 \cap C$ and $\mathcal{O}_2 \cap C$ both nonempty and with $C \subset \mathcal{O}_1 \cup \mathcal{O}_2$. Otherwise, C is said to be connected.

(j) A function $f : \mathcal{T} \to \mathcal{T}'$ is said to be continuous if $f^{-1}(\mathcal{O}') \in \mathcal{T}$ for all $\mathcal{O}' \in \mathcal{T}'$.

(k) A function $f : \mathcal{T} \to \mathcal{T}'$ is said to be continuous at the point $x \in X$ if for each $\mathcal{O}' \in \mathcal{T}'$ with $y = f(x) \in \mathcal{O}'$, there is an $\mathcal{O} \in \mathcal{T}$ with $x \in \mathcal{T}$ and $\mathcal{O} \subset f^{-1}(\mathcal{O}')$ (or, equivalently, $f(\mathcal{O}) \subset \mathcal{O}'$.

Example 3

(a) If X is any set and $\mathcal{T} = \{\emptyset, X\}$, then $(X, \mathcal{T})$ is a topological space. This is called the trivial topology for X. In this topology, every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to every $x \in X$. In particular, limits are not unique. In this topology all subsets of X are compact.

(b) If X is any set and $\mathcal{T}$ is the collection of all subsets of X, then $(X, \mathcal{T})$ is a topological space. This is called the discrete topology for X. In this topology, a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ if and only if there is a natural number N such that $x_n = x$ for all $n \geq N$. In this topology, a subset of X is compact if and only if it contains only finitely many points.

(c) If X is a metric space and $\mathcal{T}$ is collection of all subsets $\mathcal{O} \subset X$ with the property that for each $x \in \mathcal{O}$ there is an $\varepsilon > 0$ such that the open ball of radius ε centred on x is contained in $\mathcal{O}$ (i.e. $\mathcal{T}$ is the collection of sets that are open in the metric space sense), then $(X, \mathcal{T})$ is a topological space.

(d) If X is any infinite set and a subset $S \subset X$ is in $\mathcal{T}$ if and only if either S is empty or the complement of S is finite, then $(X, \mathcal{T})$ is a topological space. This is called the cofinite topology for X. Under this topology a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ if and only if, for each $y \neq x$, x_n takes the value y only for finitely many $n \in \mathbb{N}$. For each sequence $\Sigma = \{x_n\}_{n\in\mathbb{N}} \subset X$, set

$$\Lambda_\Sigma = \left\{ \ x \in X \ \middle| \ x_n = x \text{ for infinitely many different } n \in \mathbb{N} \ \right\}$$

If $\Lambda_\Sigma = \emptyset$, then Σ converges to every $x \in X$. If Λ_Σ contains a single point, then Σ converges to that point. If Λ_Σ contains strictly more than one point, then Σ does not converges to anything.

Lemma 4 *Let $(X, \mathcal{T})$ be a topological space, $A, B, A_1, A_2, \cdots \subset X$ and $n \in \mathbb{N}$.*

(a) $\overline{\emptyset} = \emptyset$

(b) $\emptyset^\circ = \emptyset$

(c) $A \subset \overline{A}$

(d) $A^\circ \subset A$

(e) If A is closed, then $A = \overline{A}$.

(f) If A is open, then $A^\circ = A$.

(e) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(f) If $A \subset B$, then $A^\circ \subset B^\circ$.

(g) If $B_n = \bigcup_{i=1}^{n} A_i$, then $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$.

(h) If $B_n = \bigcap_{i=1}^{n} A_i$, then $B_n^\circ = \bigcap_{i=1}^{n} A_i^\circ$.

(i) If $B = \bigcup_{i=1}^{\infty} A_i$, then $\overline{B} \supset \bigcup_{i=1}^{\infty} \overline{A_i}$.

(j) If $B = \bigcap_{i=1}^{\infty} A_i$, then $B^\circ \subset \bigcap_{i=1}^{\infty} A_i^\circ$.

Proof: Exercise. You should also show, by examples, that the inclusions in (i) and (j) can be proper. ■

Lemma 5 *Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces. A function $f : X \to X'$ is continuous if and only if it is continuous at each point $x \in X$.*

Proof: Exercise. ■

To avoid nonuniqueness of limits, as in Examples 3.a,d, one usually deals with Hausdorff topological spaces (or Hausdorff spaces), as defined in

Definition 6 (Hausdorff) A topological space $(X, \mathcal{T})$ is said to be Hausdorff if for each pair x, y of distinct points in X there are open sets $\mathcal{O}_x, \mathcal{O}_y \in \mathcal{T}$ such that $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y$ and $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$.

Lemma 7 *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in a Hausdorff space $(X, \mathcal{T})$. If*

$$\lim_{n\to\infty} x_n = x \qquad \text{and} \qquad \lim_{n\to\infty} x_n = y$$

then $x = y$.

Proof: Exercise. ■

Proposition 8 *Let X and Y be topological spaces.*

(i) A closed subset of a compact subset of X is compact.

(ii) If K is a compact subset of a Hausdorff topological space X and $x \notin K$, then there are disjoint open sets $U, V \subset X$ with $x \in U$ and $K \subset V$.

(iii) If K is a compact subset of a Hausdorff topological space X then K is closed.

(iv) If $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K) \subset Y$ is compact.

Proof: (i) Let $K \subset X$ be compact and $A \subset K$ be closed (in X). If $\{U_i\}_{i\in\mathcal{I}}$ is an open cover of A, then $\{U_i\}_{i\in\mathcal{I}} \cup \{X \setminus A\}$ is an open cover of K. As K is compact, there is a finite subset $\mathcal{J} \subset \mathcal{I}$ such that $K \subset (X \setminus A) \cup \bigcup_{j\in\mathcal{J}} U_j$. Then $A \subset \bigcup_{j\in\mathcal{J}} U_j$. Thus A is compact.

(ii) For each $y \in K$ choose disjoint open subsets U_y, V_y such that $x \in U_y$ and $y \in V_y$. As $\{V_y\}_{y\in K}$ is an open cover of K, there is a finite subset $K' \subset K$ such that $K \subset \bigcup_{y\in K'} V_y$. Then $V = \bigcup_{y\in K'} V_y$ and $U = \bigcap_{y\in K'} U_y$ do the job.

(iii) By part (ii), for each $x \in X \setminus K$ there is an open set U_x with $x \in U_x$ and $U_x \cap K = \emptyset$. Thus $X \setminus K = \bigcup_{x\in X\setminus K} U_x$ is a union of open sets and hence is itself open.

(iv) Let $\{\mathcal{O}_j\}_{j\in\mathcal{J}}$ be an open cover of $f(K)$. Then $\{f(\mathcal{O}_j)^{-1}\}_{j\in\mathcal{J}}$ is an open cover of K. As K is compact, there is a finite set $\mathcal{I} \subset \mathcal{J}$ of indices such that $\{f(\mathcal{O}_j)^{-1}\}_{j\in\mathcal{I}}$ still covers K. Then $\{\mathcal{O}_j\}_{j\in\mathcal{I}}$ covers $f(K)$. ■

Definition 9 (Base) Let $(X, \mathcal{T})$ be a topological space.

(a) A base for $\mathcal{T}$ is a collection $\mathcal{B} \subset \mathcal{T}$ of open sets with the property that every $\mathcal{O} \in \mathcal{T}$ is a union of elements of $\mathcal{B}$. Equivalently, $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ if for each $x \in X$ and $\mathcal{O} \in \mathcal{T}$ with $x \in \mathcal{O}$, there is a $\mathcal{U} \in \mathcal{B}$ with $x \in \mathcal{U} \subset \mathcal{O}$.

(b) A base for $\mathcal{T}$ at $x \in X$ is a collection $\mathcal{B}_x \subset \mathcal{T}$ of open sets with the property that for each $\mathcal{O} \in \mathcal{T}$ with $x \in \mathcal{O}$, there is a $\mathcal{U} \in \mathcal{B}_x$ with $x \in \mathcal{U} \subset \mathcal{O}$.

Definition 10 (First Countable) A topological space satisfies the first axiom of countability if it has a countable base at each point.

Definition 11 (Second Countable) A topological space is second countable if it has a countable base.

Example 12 Every metric space is first countable. Indeed

$$\mathcal{B}_x = \left\{ B_r(x) \;\middle|\; r \in \mathbb{Q},\ r > 0 \right\}$$

is a countable base at x.

Example 13 If a metric space is separable, meaning that it has a countable dense subset, then it is second countable. Indeed, if $\{x_\ell\}$, is a countable dense subset then

$$\mathcal{B} = \left\{ \ B_r(x_\ell) \ \middle| \ \ell \in \mathbb{N}, \ r \in \mathbb{Q}, \ r > 0 \ \right\}$$

is a countable base. (Here $B_r(x_\ell)$ is the open ball of radius r centred on x_ℓ.) This is because if $\mathcal{O}$ is open and $x \in \mathcal{O}$, then, by definition, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \mathcal{O}$. So if r is any positive rational number smaller than $\frac{\varepsilon}{2}$ and ℓ is chosen so that x_ℓ is less than distance r from x, we have $x \in B_r(x_\ell) \subset B_\varepsilon(x) \subset \mathcal{O}$.

As a special case of this example, we have that $\mathbb{R}^n$ is second countable.

Lemma 14 *Let $\langle X, \mathcal{T} \rangle$ be a topological space.*

(a) If $\langle X, \mathcal{T} \rangle$ is sequentially compact, then it is countably compact.

(b) If $\langle X, \mathcal{T} \rangle$ is countably compact and first countable, then it is sequentially compact.

Proof: Exercise. ■

Definition 15 (Relative Topology) Let $(X, \mathcal{T})$ be a topological space and S a subset of X. Then $(S, \mathcal{T}_S)$ with

$$\mathcal{T}_S = \left\{ \ \mathcal{O} \cap S \ \middle| \ \mathcal{O} \in \mathcal{T} \ \right\}$$

is a topological space. $\mathcal{T}_S$ is called the relative (or induced or subspace) topology for S.

Example 16 (Relative Topology)

(a) Let X be a metric space with metric d and with the metric space topology. Then any subset $S \subset X$ can also be viewed as a metric space with metric d. The relative topology for S is again just the metric space topology for S.

(b) Let X be $\mathbb{R}$ with the usual topology. Let $S = [0, 2)$ with the relative topology. Then, viewed as a subset of S, the half open interval $[0, 1)$ is an open set and, viewed as a subset of S, the half open interval $[1, 2)$ is a closed set.

(c) If S is an open subset of X (that is $S \in \mathcal{T}$) then $A \subset S$ is open in S (i.e. $A \in \mathcal{T}_S$) if and only if it is open in X (that is $A \in \mathcal{T}$).

(d) If S is a closed subset of X (that is $X \setminus S \in \mathcal{T}$) then $A \subset S$ is closed in S (i.e. $S \setminus A \in \mathcal{T}_S$) if and only if it is closed in X (that is $X \setminus A \in \mathcal{T}$).

References

▷ Gerald B. Folland, *Real Analysis, Modern Techniques and Their Applications*, §4.

▷ H. L. Royden, *Real Analysis*, §8,9.

▷ John L. Kelley *General Topology.*