Examples of Manifolds

A manifold is a generalization of a surface. Roughly speaking, a d–dimensional manifold is a set that looks locally like $\mathbb{R}^d$. It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of $\mathbb{R}^d$. Here is a precise definition.

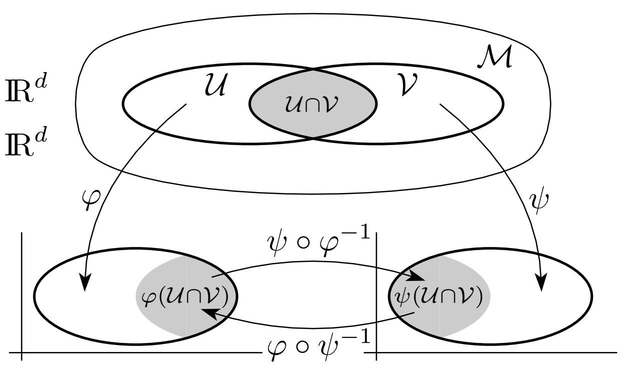
Definition 1 We now define what is meant by the statement that $\mathcal{M}$ is a d–dimensional manifold of class C^k (with $1 \le k \le \infty$ — we shall deal almost exclusively with manifolds of class C^∞).

(a) Let $\mathcal{M}$ be a Hausdorff topological space[(1)]. A *coordinate system* (or *chart* or *coordinate patch*) on $\mathcal{M}$ is a pair $(\mathcal{U}, \varphi)$ with $\mathcal{U}$ a connected open subset of $\mathcal{M}$ and φ a homeomorphism (a 1–1, onto, continuous function with continuous inverse) from $\mathcal{U}$ onto an open subset of $\mathbb{R}^d$. Think of φ as assigning coordinates to each point of $\mathcal{U}$. A coordinate system $(\mathcal{U}, \varphi)$ is called a cubic coordinate system if $\varphi(U)$ is an open cube about the origin in $\mathbb{R}^d$. (That is, if there are numbers $a_1, \cdots, a_d, b_1, \cdots, b_d > 0$ such that $\varphi(U) = \big\{\, x \in \mathbb{R}^d \,\big|\, -a_i < x_i < b_i, \ \text{ for all } 1 \le i \le d \,\big\}$.) If $m \in U$ and $\varphi(m) = 0$, then the coordinate system is said to be centred at m.

(b) A *locally Euclidean space of dimension d*, is a Hausdorff topological space for which every point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^d$.

(c) Two charts $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ are said to be *compatible* of class C^k if the transition functions

$$\psi \circ \varphi^{-1} \colon \ \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d \to \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d$$
$$\varphi \circ \psi^{-1} \colon \ \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^d$$



are C^k. That is, all partial derivatives up to order k (for C^∞, all partial derivatives of all orders) of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ exist and are continuous.

(d) An *atlas* of class C^k for a locally Euclidean space $\mathcal{M}$ is a family $\mathcal{A} = \big\{\, (\mathcal{U}_i, \varphi_i) \,\big|\, i \in \mathcal{I} \,\big\}$ of coordinate systems on $\mathcal{M}$ such that $\bigcup_{i \in \mathcal{I}} \mathcal{U}_i = \mathcal{M}$ and such that every pair of charts

(1) If you don't know what this means, substitute "metric space" for "Hausdorff topological space" and read the notes "A Little Point Set Topology".

in $\mathcal{A}$ is compatible of class C^k. The index set $\mathcal{I}$ is completely arbitrary. It could consist of just a single index. It could consist of uncountably many indices. An atlas $\mathcal{A}$ is called *maximal* of class C^k if every chart $(\mathcal{U}, \varphi)$ on $\mathcal{M}$ that is compatible of class C^k with every chart of $\mathcal{A}$ is itself in $\mathcal{A}$. A maximal atlas of class C^k is also called a *differentiable structure of class* C^k.

(e) An *d-dimensional manifold of class* C^k is a pair $(\mathcal{M}, \mathcal{A})$ with $\mathcal{M}$ a d–dimensional, second countable, locally Euclidean space and $\mathcal{A}$ a differentiable structure of class C^k. "Second countable" means that $\mathcal{M}$ has a countable base. A *base* is a collection $\mathcal{B}$ of open subsets of $\mathcal{M}$ with the property that every open subset of $\mathcal{M}$ is a union of elements of $\mathcal{B}$. A metric space that contains a countable dense subset (the metric space is then said to be separable) is automatically second countable. The countable base is the set of all open balls $B_r(x) = \{ \, y \in \mathcal{M} \bigm| d(x,y) < r \, \}$ with centre x in the countable dense subset and radius r a positive rational number.

Problem 1 Let $\mathcal{A}$ be an atlas for the Hausdorff space $\mathcal{M}$. Prove that there is a unique maximal atlas for $\mathcal{M}$ that contains $\mathcal{A}$.

Problem 2 Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of a Hausdorff space $\mathcal{M}$. Let φ be a homeomorphism from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$ and ψ be a homeomorphism from $\mathcal{V}$ to an open subset of $\mathbb{R}^m$. Prove that if $\mathcal{U} \cap \mathcal{V}$ is nonempty and

$$\psi \circ \varphi^{-1}:\ \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \to \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m$$
$$\varphi \circ \psi^{-1}:\ \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \to \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n$$

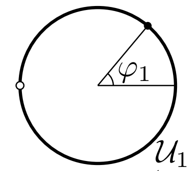
are C^1, then $m = n$.

Thanks to Problem 1, it suffices to supply any, not necessarily maximal, atlas for a second countable Hausdorff space to turn it into a manifold. We do exactly that in each of the following examples.

Example 2 (Open Subset of $\mathbb{R}^d$) $\mathbb{R}^d$ is a metric space and hence is a Hausdorff topological space. It is second countable because the the set of all open balls with rational radii and rational centres is a countable base. Let $\mathbb{1}_d$ be the identity map on $\mathbb{R}^d$. Then $\big\{(\mathbb{R}^d, \mathbb{1}_d)\big\}$ is an atlas for $\mathbb{R}^d$. Indeed, if $\mathcal{U}$ is any nonempty, open subset of $\mathbb{R}^d$, then $\big\{(\mathcal{U}, \mathbb{1}_d)\big\}$ is an atlas for $\mathcal{U}$. So every open subset of $\mathbb{R}^d$ is naturally a C^∞ manifold.

Example 3 (The Circle) The circle $S^1 = \{ \, (x,y) \in \mathbb{R}^2 \bigm| x^2 + y^2 = 1 \, \}$ is a manifold of dimension one when equipped with, for example, the atlas $\mathcal{A} = \{(\mathcal{U}_1, \varphi_1),\ (\mathcal{U}_2, \varphi_2)\}$ where

$$\mathcal{U}_1 = S^1 \setminus \{(-1,0)\} \quad \varphi_1(x,y) = \arctan\tfrac{y}{x} \text{ with } -\pi < \varphi_1(x,y) < \pi$$
$$\mathcal{U}_2 = S^1 \setminus \{(1,0)\} \quad \varphi_2(x,y) = \arctan\tfrac{y}{x} \text{ with } 0 < \varphi_2(x,y) < 2\pi$$



My use of $\arctan\frac{y}{x}$ here is pretty sloppy. To define φ_1 carefully, we can say that $\varphi_1(x,y)$ is the unique $-\pi < \theta < \pi$ such that $(x,y) = (\cos\theta, \sin\theta)$. To verify that these two charts are compatible, we first determine the domain intersection $\mathcal{U}_1 \cap \mathcal{U}_2 = S^1 \setminus \{(-1,0),(1,0)\}$ and then the ranges $\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) = (-\pi,0) \cup (0,\pi)$ and $\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) = (0,\pi) \cup (\pi,2\pi)$ and finally, we check that

$$\varphi_2 \circ \varphi_1^{-1}(\theta) = \begin{Bmatrix} \theta & \text{if } 0 < \theta < \pi \\ \theta + 2\pi & \text{if } -\pi < \theta < 0 \end{Bmatrix} \qquad \varphi_1 \circ \varphi_2^{-1}(\theta) = \begin{Bmatrix} \theta & \text{if } 0 < \theta < \pi \\ \theta - 2\pi & \text{if } \pi < \theta < 2\pi \end{Bmatrix}$$

are indeed C^∞.

Example 4 (The d–Sphere) The d–sphere

$$S^d = \left\{ \mathbf{x} = (x_1, \cdots, x_{d+1}) \in \mathbb{R}^{d+1} \ \middle|\ x_1^2 + \cdots + x_{d+1}^2 = 1 \right\}$$

is a manifold of dimension d when equipped with the atlas

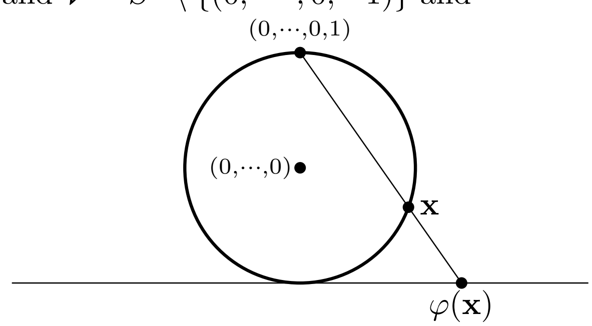
$$\mathcal{A}_1 = \left\{ (\mathcal{U}_i, \varphi_i),\ (\mathcal{V}_i, \psi_i) \ \middle|\ 1 \le i \le d+1 \right\}$$

where, for each $1 \le i \le d+1$,

$$\mathcal{U}_i = \left\{ (x_1, \cdots, x_{d+1}) \in S^d \ \middle|\ x_i > 0 \right\} \quad \varphi_i(x_1, \cdots, x_{d+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{d+1})$$
$$\mathcal{V}_i = \left\{ (x_1, \cdots, x_{d+1}) \in S^d \ \middle|\ x_i < 0 \right\} \quad \psi_i(x_1, \cdots, x_{d+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{d+1})$$

So both φ_i and ψ_i just discard the coordinate x_i. They project onto $\mathbb{R}^d$, viewed as the hyperplane $x_i = 0$. Another possible atlas, compatible with $\mathcal{A}_1$, is $\mathcal{A}_2 = \left\{ \big(\mathcal{U}, \varphi\big)\ ,\ \big(\mathcal{V}, \psi\big) \right\}$ where the domains $\mathcal{U} = S^d \setminus \{(0, \cdots, 0, 1)\}$ and $\mathcal{V} = S^d \setminus \{(0, \cdots, 0, -1)\}$ and

$$\varphi(x_1, \cdots, x_{d+1}) = \left(\tfrac{2x_1}{1-x_{d+1}}, \cdots, \tfrac{2x_d}{1-x_{d+1}}\right)$$
$$\psi(x_1, \cdots, x_{d+1}) = \left(\tfrac{2x_1}{1+x_{d+1}}, \cdots, \tfrac{2x_d}{1+x_{d+1}}\right)$$



are the stereographic projections from the north and south poles, respectively. Both φ and ψ have range $\mathbb{R}^d$. So we can think of S^d as $\mathbb{R}^d$ plus an additional single "point at infinity".

Problem 3 In this problem we use the notation of Example 4.

(a) Prove that $\mathcal{A}_1$ is an atlas for S^d.

(b) Prove that $\mathcal{A}_2$ is an atlas for S^d.

Example 5 (Surfaces) Any smooth n–dimensional surface in $\mathbb{R}^{n+m}$ is an n–dimensional manifold. Roughly speaking, a subset of $\mathbb{R}^{n+m}$ is an n–dimensional surface if, locally, m of the $m+n$ coordinates of points on the surface are determined by the other n coordinates in a C^∞ way. For example, the unit circle S^1 is a one dimensional surface in $\mathbb{R}^2$. Near $(0,1)$ a point $(x,y)\in\mathbb{R}^2$ is on S^1 if and only if $y=\sqrt{1-x^2}$, and near $(-1,0)$, (x,y) is on S^1 if and only if $x=-\sqrt{1-y^2}$.

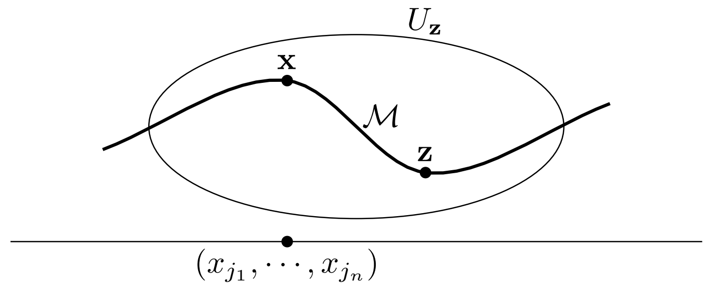
The precise definition is that $\mathcal{M}$ is an n–dimensional surface in $\mathbb{R}^{n+m}$ if $\mathcal{M}$ is a subset of $\mathbb{R}^{n+m}$ with the property that for each $\mathbf{z}=(z_1,\cdots,z_{n+m})\in\mathcal{M}$, there are

- ◦ a neighbourhood $U_\mathbf{z}$ of $\mathbf{z}$ in $\mathbb{R}^{n+m}$
- ◦ n integers $1\le j_1<j_2<\cdots<j_n\le n+m$
- ◦ and m C^∞ functions $f_k(x_{j_1},\cdots,x_{j_n})$, $k\in\{1,\cdots,n+m\}\setminus\{j_1,\cdots,j_n\}$

such that the point $\mathbf{x}=(x_1,\cdots,x_{n+m})\in U_\mathbf{z}$ is in $\mathcal{M}$ if and only if $x_k=f_k(x_{j_1},\cdots,x_{j_n})$ for all $k\in\{1,\cdots,n+m\}\setminus\{j_1,\cdots,j_n\}$. That is, we may express the part of $\mathcal{M}$ that is near $\mathbf{z}$ as

$$\begin{aligned} x_{i_1} &= f_{i_1}\big(x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}\big)\\ x_{i_2} &= f_{i_2}\big(x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}\big)\\ &\ \ \vdots\\ x_{i_m} &= f_{i_m}\big(x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}\big) \end{aligned}$$

where $\big\{i_1,\cdots,i_m\big\}=\{1,\cdots,n+m\}\setminus\{j_1,\cdots,j_n\}$



for some C^∞ functions $f_1,\ \cdots,\ f_m$. We may use $x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_n}$ as coordinates for $\mathcal{M}$ in $\mathcal{M}\cap U_\mathbf{z}$. Of course, an atlas is $\mathcal{A}=\big\{\ (U_\mathbf{z}\cap\mathcal{M},\varphi_\mathbf{z})\ \big|\ \mathbf{z}\in M\ \big\}$, with $\varphi_\mathbf{z}(\mathbf{x})=(x_{j_1},\cdots,x_{j_n})$. Equivalently, $\mathcal{M}$ is an n–dimensional surface in $\mathbb{R}^{n+m}$, if, for each $\mathbf{z}\in\mathcal{M}$, there are

- ◦ a neighbourhood $U_\mathbf{z}$ of $\mathbf{z}$ in $\mathbb{R}^{n+m}$
- ◦ and m C^∞ functions $g_k:U_\mathbf{z}\to\mathbb{R}$, with the vectors $\big\{\ \boldsymbol{\nabla} g_k(\mathbf{z})\ \big|\ 1\le k\le m\ \big\}$ linearly independent

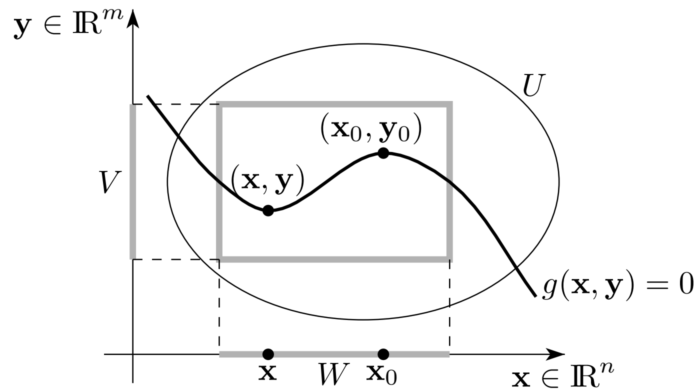
such that the point $\mathbf{x}\in U_\mathbf{z}$ is in $\mathcal{M}$ if and only if $g_k(\mathbf{x})=0$ for all $1\le k\le m$. To get from the implicit equations for $\mathcal{M}$ given by the g_k's to the explicit equations for $\mathcal{M}$ given by the f_k's one need only invoke (possible after renumbering the components of $\mathbf{x}$) the

Implicit Function Theorem
Let $m,n\in\mathbb{N}$ and let $U\subset\mathbb{R}^{n+m}$ be an open set. Let $\mathbf{g}:U\to\mathbb{R}^m$ be C^∞ with $\mathbf{g}(\mathbf{x}_0,\mathbf{y}_0)=0$ for some $\mathbf{x}_0\in\mathbb{R}^n$, $\mathbf{y}_0\in\mathbb{R}^m$ with $(\mathbf{x}_0,\mathbf{y}_0)\in U$. Assume that

$\det\left[\frac{\partial g_i}{\partial y_j}(\mathbf{x}_0,\mathbf{y}_0)\right]_{1\le i,j\le m}\ne 0$. Then there exist open sets $V\subset\mathbb{R}^m$ and $W\subset\mathbb{R}^n$ with $\mathbf{x}_0\in W$ and $\mathbf{y}_0\in V$ such that

$$\text{for each } \mathbf{x}\in W, \text{ there is a unique } \mathbf{y}\in V \text{ with } \mathbf{g}(\mathbf{x},\mathbf{y})=0.$$

If the $\mathbf{y}$ above is denoted $\mathbf{f}(\mathbf{x})$, then $\mathbf{f}:W\to\mathbb{R}^m$ is C^∞, $\mathbf{f}(\mathbf{x}_0)=\mathbf{y}_0$ and $\mathbf{g}\big(\mathbf{x},\mathbf{f}(\mathbf{x})\big)=0$ for all $\mathbf{x}\in W$.



The d–sphere S^d is the d–dimensional surface in $\mathbb{R}^{d+1}$ given implicitly by the equation $g(x_1,\cdots,x_{d+1})=x_1^2+\cdots+x_{d+1}^2-1=0$. In a neighbourhood of the north pole (for example, the northern hemisphere), S^d is given explicitly by the equation $x_{d+1}=\sqrt{x_1^2+\cdots+x_d^2}$. If you think of the set of all 3×3 real matrices as $\mathbb{R}^9$ (because a 3×3 matrix has 9 matrix elements) then

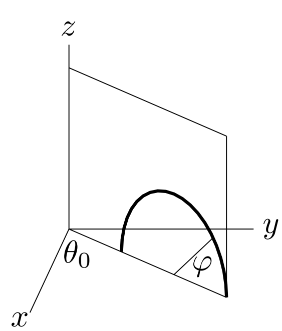
$$SO(3)=\left\{\ 3\times 3 \text{ real matrices } R \;\middle|\; R^tR=\mathbb{1},\ \det R=1\ \right\}$$

is a 3–dimensional surface in $\mathbb{R}^9$. We shall look at it more closely in Example 7, below. $SO(3)$ is the group of all rotations about the origin in $\mathbb{R}^3$ and is also the set of all orientations of a rigid body with one point held fixed.

Example 6 (A Torus) The torus T^2 is the two dimensional surface

$$T^2=\left\{\ (x,y,z)\in\mathbb{R}^3 \;\middle|\; \left(\sqrt{x^2+y^2}-1\right)^2+z^2=\tfrac14\ \right\}$$

in $\mathbb{R}^3$. In cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$, the equation of the torus is $(r-1)^2+z^2=\frac14$. Fix any θ, say θ_0. Recall that the set of all points in



$\mathbb{R}^3$ that have $\theta = \theta_0$ is like one page in an open book. It is a half–plane that starts at the z axis. The intersection of the torus with that half plane is a circle of radius $\frac{1}{2}$ centred on $r = 1$, $z = 0$. As φ runs from 0 to 2π, the point $r = 1 + \frac{1}{2}\cos\varphi$, $z = \frac{1}{2}\sin\varphi$, $\theta = \theta_0$ runs over that circle. If we now run θ from 0 to 2π, the circle on the page sweeps out the whole torus. So, as φ runs from 0 to 2π and θ runs from 0 to 2π, the point $(x, y, z) = \big((1 + \frac{1}{2}\cos\varphi)\cos\theta, (1 + \frac{1}{2}\cos\varphi)\sin\theta, \frac{1}{2}\sin\varphi\big)$ runs over the whole torus. So we may build coordinate patches for T^2 using θ and φ (with ranges $(0, 2\pi)$ or $(-\pi, \pi)$) as coordinates.

Example 7 **($O(3)$, $SO(3)$)** As a special case of Example 5 we have the groups

$$\begin{aligned} SO(3) &= \left\{\ 3\times 3 \text{ real matrices } R \;\middle|\; R^tR = \mathbb{1}_3,\ \det R = 1\ \right\} \\ O(3) &= \left\{\ 3\times 3 \text{ real matrices } R \;\middle|\; R^tR = \mathbb{1}_3\ \right\} \end{aligned}$$

of rotations and rotations/reflections in $\mathbb{R}^3$. (Rotations and reflections are the angle and length preserving linear maps. In classical mechanics, $SO(3)$ is the set of all possible configurations of rigid body with one point held fixed.) We can identify the set of all 3×3 real matrices with $\mathbb{R}^9$, because a 3×3 matrix has 9 matrix elements. The restriction that

$$R = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \in O(3)$$

is given implicitly by the following six equations.

$$\begin{aligned} \big(R^tR\big)_{1,1} &= a_1^2 + a_2^2 + a_3^2 = 1 && \text{i.e. } |\mathbf{a}| = 1 \\ \big(R^tR\big)_{2,2} &= b_1^2 + b_2^2 + b_3^2 = 1 && \text{i.e. } |\mathbf{b}| = 1 \\ \big(R^tR\big)_{3,3} &= c_1^2 + c_2^2 + c_3^2 = 1 && \text{i.e. } |\mathbf{c}| = 1 \\ \big(R^tR\big)_{1,2} = \big(R^tR\big)_{2,1} &= a_1b_1 + a_2b_2 + a_3b_3 = 0 && \text{i.e. } \mathbf{a}\perp\mathbf{b} \\ \big(R^tR\big)_{1,3} = \big(R^tR\big)_{3,1} &= a_1c_1 + a_2c_2 + a_3c_3 = 0 && \text{i.e. } \mathbf{a}\perp\mathbf{c} \\ \big(R^tR\big)_{2,3} = \big(R^tR\big)_{3,2} &= b_1c_1 + b_2c_2 + b_3c_3 = 0 && \text{i.e. } \mathbf{b}\perp\mathbf{c} \end{aligned} \tag{1}$$

We can verify the independence conditions of Example 5 (that the gradients of the left hand sides are independent) directly. See Problems 5 and 6, below. Or we can argue geometrically. In a neighbourhood of any fixed element, $\tilde{R}$, of $SO(3)$, we may use two of the three **a**–components as coordinates. (In fact we may use any two **a**–coordinates whose magnitude at $\tilde{R}$ is not one.) Once two components of **a** have been chosen, the third **a**–component is determined up to a sign by the requirement that $|\mathbf{a}| = 1$. The sign is chosen so as to remain in the neighbourhood. Once **a** has been chosen, the set $\left\{\ \mathbf{b}\in\mathbb{R}^3 \;\middle|\; \mathbf{b}\perp\mathbf{a}\ \right\}$ is a plane through the origin so that $\left\{\ \mathbf{b}\in\mathbb{R}^3 \;\middle|\; \mathbf{b}\perp\mathbf{a},\ |\mathbf{b}| = 1\ \right\}$ is the intersection of that plane with the unit sphere. So $\mathbf{b}$ lies on a great circle of the unit sphere. Thus $\mathbf{b}$ is determined up to a single rotation angle by the requirements that $\mathbf{b}\perp\mathbf{a}$ and $|\mathbf{b}|=1$. That rotation angle is the third coordinate. Once $\mathbf{a}$ and $\mathbf{b}$ have been chosen, the set $\{\, \mathbf{c}\in\mathbb{R}^3 \bigm| \mathbf{c}\perp\mathbf{a},\ \mathbf{c}\perp\mathbf{b} \,\}$ is a line through the origin. So $\mathbf{c}$ is determined up to a sign by the requirements that $\mathbf{c}\perp\mathbf{a},\mathbf{b}$ and $|\mathbf{c}|=1$. Again, the sign is chosen so as to remain in the neighbourhood. So $O(3)$ is a manifold of dimension 3. Any element of $O(3)$ automatically obeys

$$\big(\det R\big)^2 = \det R^tR = \det \mathbb{1}_3 = 1 \implies \det R = \pm 1$$

So $SO(3)$ is just one of the two connected components of $O(3)$. It is an important example of a Lie group, which is, by definition, a C^∞ manifold that is also a group with the operations of multiplication and taking inverses continuous.

Problem 4 Let $R\in O(3)$.

(a) Prove that if λ is an eigenvalue of R, then $|\lambda|=1$ and $\bar\lambda$ is an eigenvalue of R.

(b) Prove that at least one eigenvalue of R is either $+1$ or -1.

(c) Prove that the columns of R are mutually perpendicular and are each of unit length.

(d) Prove that R is either a rotation, a reflection or a composition of a rotation and a reflection.

Problem 5 Denote by $g_1,\cdots,g_6$ the left hand sides of (1). Prove that the gradients of $g_1,\cdots,g_6$, evaluated at any $R\in O(3)$, are linearly independent.

Problem 6 Use the implicit function theorem to prove that for each $1\le i,j\le 3$, the (i,j) matrix element, a_{ij}, of matrices $R=\big[a_{ij}\big]_{1\le i,j\le 3}$ in a neighbourhood of $\mathbb{1}$ in $SO(3)$, is a C^∞ function of the matrix elements a_{21}, a_{31} and a_{32}.

Example 8 **(More Tori)** Define an equivalence relation on $\mathbb{R}^d$ by

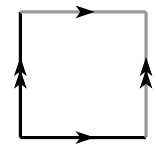
$$x\sim y \iff x-y\in\mathbb{Z}^d$$

In this example, when $x\sim y$ we want to think of x and y as two different names for the same object. The set of all possible names for the object whose name is also x is $[x]=\{\, y\in\mathbb{R}^d \bigm| y\sim x \,\}$ and is called the equivalence class of $x\in\mathbb{R}^d$. The set of equivalence classes is denoted $\mathbb{R}^d/\mathbb{Z}^d=\{\, [x] \bigm| x\in\mathbb{R}^d \,\}$. Each equivalence class $[x]$ contains exactly one representative $\tilde x\in[x]$ obeying $0\le\tilde x_j<1$ for each $1\le j\le d$. So we can also think of $\mathbb{R}^d/\mathbb{Z}^d$ as being

$$\{\, x\in\mathbb{R}^d \bigm| 0\le x_j<1 \text{ for all } 1\le j\le d \,\}$$

But then we should also identify, for each $1 \le j \le d$, the edges

$$\begin{aligned} &\big\{\ x \in \mathbb{R}^d\ \big|\ x_j = 1,\ 0 \le x_i \le 1\ \forall\ i \ne j\ \big\} \\ \text{and } &\big\{\ x \in \mathbb{R}^n\ \big|\ x_j = 0,\ 0 \le x_i \le 1\ \forall\ i \ne j\ \big\} \end{aligned}$$



We can turn the set $\mathbb{R}^d/\mathbb{Z}^d$, which is also called a torus, into a metric space by imposing the metric

$$\rho\big([x],[y]\big) = \min\big\{\ |\tilde{x}-\tilde{y}|\ \big|\ \tilde{x} \in [x],\ \tilde{y} \in [y]\ \big\}$$

So we only need an atlas to turn the torus into a manifold. If $\mathcal{U}$ is any open subset of $\mathbb{R}^d$ with the property that no two points of $\mathcal{U}$ are equivalent (any open ball of radius at most $\frac{1}{2}$ has this property), then $[\mathcal{U}] = \big\{\ [x]\ \big|\ x \in \mathcal{U}\ \big\}$ is an open subset of $\mathbb{R}^d/\mathbb{Z}^d$ and each element of $[\mathcal{U}]$ contains a unique representative $\tilde{x} \in [x]$ that is in $\mathcal{U}$. Define

$$\begin{aligned} \Phi_{\mathcal{U}} : [\mathcal{U}] &\to \mathbb{R}^d \\ [x] &\mapsto \tilde{x} \text{ with } \tilde{x} \in [x],\ \tilde{x} \in \mathcal{U} \end{aligned}$$

Then $\{[\mathcal{U}], \Phi_{\mathcal{U}}\}$ is a chart and the set of all such charts is an atlas.

Example 9 (The Cartesian Product) If $\mathcal{M}$ is a manifold of dimension m with atlas $\mathcal{A}$ and $\mathcal{N}$ is a manifold of dimension n with atlas $\mathcal{B}$ then

$$\mathcal{M} \times \mathcal{N} = \big\{\ (x,y)\ \big|\ x \in \mathcal{M},\ y \in \mathcal{N}\ \big\}$$

is an $(m+n)$–dimensional manifold with atlas

$$\big\{\ \big(U \times V, \varphi \oplus \psi\big)\ \big|\ (U,\varphi) \in \mathcal{A},\ (V,\psi) \in \mathcal{B}\ \big\} \qquad \text{where } \varphi \oplus \psi\big((x,y)\big) = \big(\varphi(x), \psi(y)\big)$$

For example, $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, $S^1 \times \mathbb{R}$ is a cylinder, $S^1 \times S^1$ is a torus and the configuration space of a rigid body is $\mathbb{R}^3 \times SO(3)$ (with the $\mathbb{R}^3$ components giving the location of the centre of mass of the body and the $SO(3)$ components giving the orientation).

Example 10 (The Möbius Strip) We are now going to turn the set

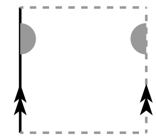
$$\mathcal{M} = [0,1) \times (-1,1) = \big\{\ (s,t)\ \big|\ 0 \le s < 1,\ -1 < t < 1\ \big\}$$

into two very different manifolds by assigning two different, incompatible, atlases. Both atlases will contain two charts with

$$\mathcal{U}_1 = \big(\tfrac{1}{8}, \tfrac{7}{8}\big) \times (-1,1) \qquad \mathcal{U}_2 = \big[0, \tfrac{1}{4}\big) \times (-1,1) \cup \big(\tfrac{3}{4}, 1\big) \times (-1,1)$$

The first atlas attaches each point $(0,t)$ on the left hand edge to the point $(1,t)$ on the right hand edge by using the coordinate functions

$$(x,y)=\psi_1(s,t)=(s,t)$$
$$(x,y)=\psi_2(s,t)=\begin{cases}(s,t) & \text{if } 0\le s<\frac14\\ (s-1,t) & \text{if } \frac34<s<1\end{cases}$$



The range of ψ_2 is

$$\psi_2\big([0,\tfrac14)\times(-1,1)\big)\ \cup\ \psi_2\big((\tfrac34,1)\times(-1,1)\big)=[0,\tfrac14)\times(-1,1)\ \cup\ \big(-\tfrac14,0\big)\times(-1,1)$$
$$=\big(-\tfrac14,\tfrac14\big)\times(-1,1)$$

The inverse map for ψ_2 is

$$(s,t)=\psi_2^{-1}(x,y)=\begin{cases}(x,y) & \text{if } 0\le x<\frac14\\ (x+1,y) & \text{if } -\frac14<x<0\end{cases}$$

The inverse image under ψ_2 of the disk $x^2+\big(y-\frac12\big)^2<\frac1{16}$ (denote it $B_{\frac14}(0,\frac12)$) is

$$\psi_2^{-1}\big(B_{\frac14}(0,\tfrac12)\cap\{x\ge0\}\big)\ \cup\ \psi_2^{-1}\big(B_{\frac14}(0,\tfrac12)\cap\{x<0\}\big)$$
$$=B_{\frac14}(0,\tfrac12)\cap\{x\ge0\}\ \cup\ \big\{\ (x+1,y)\ \big|\ (x,y)\in B_{\frac14}(0,\tfrac12),\ x<0\ \big\}$$

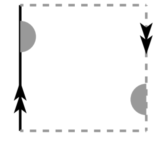
That is the union of the two shaded half disks displayed in the figure above. The union is connected in the manifold with atlas $\big\{\{\mathcal{U}_1,\psi_1\},\{\mathcal{U}_1,\psi_2\}\big\}$. This manifold may be constructed from a strip of paper by gluing the left and right hand edges together. To complete the definition of this manifold, it suffices to provide it with a metric and then verify that $\big\{(\mathcal{U}_1,\psi_1)\,,\,(\mathcal{U}_1,\psi_2)\big\}$ really is an atlas and, in particular, that ψ_2 and its inverse are continuous. The metric (similar to the metric of Example 8)

$$\rho_\psi\big((s,t)\,,\,(s',t')\big)=\min\big\{|(s-s',t-t')|\,,\,|(s-s'+1,t-t')|\,,\,|(s-s'-1,t-t')|\big\}$$

works.

The second atlas attaches each point $(0,t)$ on the left hand edge to the point $(1,-t)$ on the right hand edge by using the coordinate functions

$$(x,y)=\varphi_1(s,t)=(s,t)$$
$$(x,y)=\varphi_2(s,t)=\begin{cases}(s,t) & \text{if } 0\le s<\frac14\\ (s-1,-t) & \text{if } \frac34<s<1\end{cases}$$



The range of φ_2 is $\big(-\frac14,\frac14\big)\times(-1,1)$, the same as the range of ψ_2. The inverse map for φ_2 is

$$(s,t)=\varphi_2^{-1}(x,y)=\begin{cases}(x,y) & \text{if } 0\le x<\frac14\\ (x+1,-y) & \text{if } -\frac14<x<0\end{cases}$$

The union of the two shaded half disks in the figure above is the inverse image under φ_2 of the disk $x^2 + \big(y - \frac{1}{2}\big)^2 < \frac{1}{16}$. That union is connected in the manifold with atlas $\big\{\{\mathcal{U}_1, \varphi_1\}, \{\mathcal{U}_1, \varphi_2\}\big\}$. This manifold may be constructed from a strip of paper by gluing the left and right hand edges together, after putting a half twist in the strip. It is called a Möbius strip. It has metric

$$\rho_\psi\big((s,t)\,,\,(s',t')\big) = \min\big\{|(s-s',t-t')|\,,\,|(s-s'+1,t+t')|\,,\,|(s-s'-1,t+t')|\big\}$$

Problem 7 Prove that the two charts $(\mathcal{U}_2, \varphi_2)$ and $(\mathcal{U}_2, \psi_2)$ of Example 10 are not compatible.

Example 11 (Projective n–space, $\mathbb{P}^n$) The projective n–space, $\mathbb{P}^n$, is the set of all lines through the origin in $\mathbb{R}^{n+1}$. If $\vec{x} \in \mathbb{R}^{n+1}$ is nonzero, then there is a unique line $L_{\vec{x}}$ through the origin in $\mathbb{R}^{n+1}$ that contains $\vec{x}$. Namely $L_{\vec{x}} = \big\{\ \lambda\vec{x}\ \big|\ \lambda \in \mathbb{R}\ \big\}$. If $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero, then $L_{\vec{x}} = L_{\vec{y}}$ if and only if there is a $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\vec{y} = \lambda\vec{x}$. One choice of atlas for $\mathbb{P}^n$ is $\mathcal{A} = \big\{\ (U_i, \varphi_i)\ \big|\ 1 \le i \le n+1\ \big\}$ with

$$U_i = \big\{\ L_{\vec{x}}\ \big|\ \vec{x} \in \mathbb{R}^{n+1},\ x_i \neq 0\ \big\} \qquad \varphi(L_{\vec{x}}) = \big(\tfrac{x_1}{x_i}, \cdots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \cdots, \tfrac{x_{n+1}}{x_i}\big) \in \mathbb{R}^n$$

Observe that if φ_i is well–defined, because if $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero and $L_{\vec{x}} = L_{\vec{y}}$, then, for each $1 \le i \le n+1$, either both x_i and y_i are zero or both x_i and y_i are nonzero and in the latter case

$$\big(\tfrac{x_1}{x_i}, \cdots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \cdots, \tfrac{x_{n+1}}{x_i}\big) = \big(\tfrac{y_1}{y_i}, \cdots, \tfrac{y_{i-1}}{y_i}, \tfrac{y_{i+1}}{y_i}, \cdots, \tfrac{y_{n+1}}{y_i}\big)$$

Each line through the origin in $\mathbb{R}^{n+1}$ intersects the unit sphere $S^n = \big\{\ \vec{x} \in \mathbb{R}^{n+1}\ \big|\ |\vec{x}| = 1\ \big\}$ in exactly two points and the two points are antipodal (i.e. $\vec{x}$ and $-\vec{x}$). So you can think of $\mathbb{P}^n$ as S^n but with antipodal points identified:

$$\mathbb{P}^{n+1} = \big\{\ \{\vec{x}, -\vec{x}\}\ \big|\ \vec{x} \in S^n\ \big\}$$

Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is not horizontal (i.e. with $x_{n+1} \neq 0$) intersects the northern hemisphere $\big\{\ \vec{x} \in \mathbb{R}^{n+1}\ \big|\ |\vec{x}| = 1,\ x_{n+1} \ge 0\ \big\}$ in exactly one point. Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is horizontal (i.e. with $x_{n+1} = 0$) intersects the northern hemisphere in exactly two points and the two points are antipodal. By ignoring x_{n+1}, you can think of the northern hemisphere as the closed unit disk $\big\{\ \mathbf{x} \in \mathbb{R}^n\ \big|\ |\mathbf{x}| \le 1\ \big\}$ in $\mathbb{R}^n$. So you can think of $\mathbb{P}^n$ as the closed unit ball in $\mathbb{R}^n$ but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified.

In the case of three dimensions, you can also think of $SO(3)$ as being the closed unit disk $\{\ \mathbf{x} \in \mathbb{R}^3 \ \big|\ |\mathbf{x}| \le 1\ \} \subset \mathbb{R}^3$ but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified. This is because, geometrically, each element of $SO(3)$ is a matrix which implements a rotation by some angle about some axis through the origin in $\mathbb{R}^3$. We can associate each $\omega\hat{\Omega} \in \mathbb{R}^3$, where $\hat{\Omega}$ is a unit vector and $\omega \in \mathbb{R}$, with the rotation by an angle $\pi\omega$ about the axis $\hat{\Omega}$. But then any two ω's that differ by an even integer give the same rotation. So the set of all rotations is associated with $\{\ \omega\hat{\Omega} \ \big|\ |\omega| \le 1,\ \hat{\Omega} \in \mathbb{R}^3,\ |\hat{\Omega}| = 1\ \}$ but with $1\hat{\Omega}$ and $-1\hat{\Omega}$ identified. Thus $SO(3)$ and $\mathbb{P}^3$ are diffeomorphic, where

Definition 12

(a) A function f from a manifold $\mathcal{M}$ to a manifold $\mathcal{N}$ (it is traditional to omit the atlas from the notation) is said to be C^∞ at $m \in \mathcal{M}$ if there exists a chart $(\mathcal{U}, \varphi)$ for $\mathcal{M}$ and a chart $(\mathcal{V}, \psi)$ for $\mathcal{N}$ such that $m \in \mathcal{U}$, $f(m) \in \mathcal{V}$ and $\psi \circ f \circ \varphi^{-1}$ is C^∞ at $\varphi(m)$.

(b) Two manifolds $\mathcal{M}$ and $\mathcal{N}$ are *diffeomorphic* if there exists a function $f : \mathcal{M} \to \mathcal{N}$ that is 1–1 and onto with f and f^{-1} C^∞ everywhere. Then you should think of $\mathcal{M}$ and $\mathcal{N}$ as the same manifold with m and $f(m)$ being two different names for the same point, for each $m \in \mathcal{M}$.

Problem 8 Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds. Prove that $f : \mathcal{M} \to \mathcal{N}$ is C^∞ at $m \in \mathcal{M}$ if and only if $\psi \circ f \circ \phi^{-1}$ is C^∞ at $\phi(m)$ for every chart $(\mathcal{U}, \phi)$ for $\mathcal{M}$ with $m \in \mathcal{U}$ and every chart $(\mathcal{V}, \psi)$ for $\mathcal{N}$ with $f(m) \in \mathcal{V}$.

Problem 9 Prove that $\mathbb{R}^n$ is diffeomorphic to $\{\ \mathbf{x} \in \mathbb{R}^n \ \big|\ \sum_{i=1}^n x_i^2 < 1\ \}$.

Problem 10 Prove that $\mathbb{R}^n$ is not diffeomorphic to S^n.

Problem 11 Outline an argument to prove that the disk $\{\ \mathbf{x} \in \mathbb{R}^2 \ \big|\ x^2 + y^2 < 2\ \}$ is not diffeomorphic to the annulus $\{\ \mathbf{x} \in \mathbb{R}^2 \ \big|\ 1 < x^2 + y^2 < 2\ \}$.

Problem 12 In this problem $G = SO(3)$.

a) Fix any $a \in G$. Denote by $I = \{\ (i, j) \in \mathbb{N}^2 \ \big|\ 1 \le i \le 3,\ 1 \le j \le 3\ \}$ the set of indices for the matrix elements of the matrices in G. Prove that there exist $\alpha, \beta, \gamma \in I$ such that every matrix element $g_\delta,\ \delta \in I$ is a C^∞ function of $g_\alpha,\ g_\beta,\ g_\gamma$ for matrices $g \in G$ in a neighbourhood of a.

b) Prove that a curve $q : (c, d) \to G$ is C^∞ if and only if every matrix element $q(t)_{i,j}$ is C^∞.

c) Prove that matrix multiplication $(a, b) \mapsto ab$ is a C^∞ function from $G \times G$ to G.

d) Prove that the inverse function $a \mapsto a^{-1}$ is a C^∞ function from G to G.

Example 13 You might think that the Hausdorff requirement that we included in the definition of a manifold is superfluous – that it is a consequence of the requirement that every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^d$. Here is an example that shows otherwise. It satisfies all of the requirements of a manifold except one — it is not Hausdorff.

To start, we just define the set

$$M = (0,1) \cup \big\{\ (x,\text{red})\ \big|\ 1 \le x < 2\ \big\} \cup \big\{\ (y,\text{yellow})\ \big|\ 1 \le y < 2\ \big\}$$

(so that M contains two distinct copies of the interval $[1,2)$ together with one copy of the interval $(0,1)$).

Next, we endow M with a topology. We give the subset

$$M_r = (0,1) \cup \big\{\ (x,\text{red})\ \big|\ 1 \le x < 2\ \big\}$$

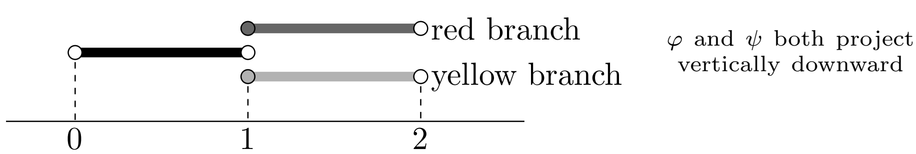
the usual topology of the real interval $(0,2)$. We also give the subset

$$M_y = (0,1) \cup \big\{\ (x,\text{yellow})\ \big|\ 1 \le x < 2\ \big\}$$

the usual topology of the real interval $(0,2)$. Then we define a subset $S \subset M$ to be open if and only if $S \cap M_r$ and $S \cap M_y$ are open. That is, $S \subset M$ is open if and only if $\big\{\ x \in (0,1)\ \big|\ x \in S\ \big\} \cup \big\{\ x \in [1,2)\ \big|\ (x,\text{red}) \in S\ \big\}$ and $\big\{\ x \in (0,1)\ \big|\ x \in S\ \big\} \cup \big\{\ x \in [1,2)\ \big|\ (x,\text{yellow}) \in S\ \big\}$ are both open subsets of $(0,2)$. This topology is not Hausdorff, because any two open sets U_1 and U_2 with $(1,\text{red}) \in U_1$ and $(1,\text{yellow}) \in U_2$ both necessarily contain $(1-\varepsilon,1)$ for some $\varepsilon > 0$ and hence necessarily intersect.

We may nonetheless give M an atlas consisting of the two charts (U,φ) and (V,ψ) where

$$\begin{aligned} U = M_r &= (0,1) \cup \big\{\ (x,\text{red})\ \big|\ 1 \le x < 2\ \big\} \\ &\qquad \varphi(x) = x \text{ for all } x \in (0,1),\ \varphi\big((x,\text{red})\big) = x \text{ for all } x \in [1,2) \\ V = M_y &= (0,1) \cup \big\{\ (y,\text{yellow})\ \big|\ 1 \le y < 2\ \big\} \\ &\qquad \psi(y) = y \text{ for all } y \in (0,1),\ \psi\big((y,\text{yellow})\big) = y \text{ for all } y \in [1,2) \end{aligned}$$



These two charts are both homeomorphisms onto $(0,2)$ and are compatible since

$$U \cap V = (0,1) \qquad \varphi \circ \psi^{-1}(y) = y \text{ and } \psi \circ \varphi^{-1}(x) = x \text{ for all } x,y \in (0,1)$$

Example 14 Here is an example that satisfies all of the requirements of a manifold except for second countability. It is called the long line. By way of motivation we reformulate the definition of the real line $\mathbb{R}$ as a manifold (but we'll call it $\tilde{\mathbb{R}}$ now), pretending that we do not already know what $\mathbb{R}$ is but that we do know what the finite intervals $[0,1)$ and (a,b) are. Define, for each integer $\ell \in \mathbb{Z}$, the set of pairs

$$I_\ell = \left\{ \ (\ell, x) \ \middle| \ x \in [0,1) \ \right\}$$

As a set, we define $\tilde{\mathbb{R}}$ to be $\bigcup_{\ell\in\mathbb{Z}} I_\ell$. (Of course, I am thinking of (ℓ, x) as another name for $\ell + x$.) That is, as a set, $\tilde{\mathbb{R}}$ is the union of countably many copies of the half open interval $[0,1)$. We may define an ordering on $\tilde{\mathbb{R}}$ by

$$(\ell, x) < (\ell', x') \iff \ell < \ell' \text{ or } \ell = \ell',\ x < x'$$

Next we introduce a topology on $\tilde{\mathbb{R}}$ by defining an open interval to be a subset of $\tilde{\mathbb{R}}$ of the form $(r,s) = \left\{ \ (\ell,x) \in \tilde{\mathbb{R}} \ \middle| \ r < (\ell,x) < s \ \right\}$ for some $r, s \in \tilde{\mathbb{R}}$ and defining a subset of $\tilde{\mathbb{R}}$ to be open if it is a union of open intervals. Finally, we introduce the atlas $\mathcal{A} = \left\{ \ (\mathcal{U}_{r,s}, \varphi_{r,s}) \ \middle| \ r, s \in \tilde{\mathbb{R}},\ r < s \ \right\}$ with

$$\begin{aligned} \mathcal{U}_{r,s} &= (r,s) \\ \varphi_{r,s}\big((\ell,x)\big) &= \ell + x \end{aligned}$$

We now define the long line $\mathbb{L}$ by repeating the above construction, but with the integers $\mathbb{Z}$ replaced by another set that we'll denote $\mathbb{Y}$. We start by replacing the natural numbers $\mathbb{N}$ with a set, $\mathbb{W}$, called the first uncountable ordinal. It is characterized by the conditions that

- $\mathbb{W}$ is totally ordered. This means that $\mathbb{W}$ is equipped with a binary relation $\le$ (i.e. a subset of $\mathbb{W} \times \mathbb{W}$) such that the following statements hold for all $a, b, c \in \mathbb{W}$.
  - If $a \le b$ and $b \le a$ then $a = b$ (antisymmetry).
  - If $a \le b$ and $b \le c$ then $a \le c$ (transitivity).
  - Either $a \le b$ or $b \le a$ (totality).
- $\mathbb{W}$ is a well–ordered set. This means that every nonempty subset of $\mathbb{W}$ has a least element.
- $\mathbb{W}$ is not countable.
- Whenever $a \in \mathbb{W}$ then $\left\{ \ i \in \mathbb{W} \ \middle| \ i \le a \ \right\}$ is countable.

A proof of the existence of the first uncountable ordinal, as well as more details on the construction of $\mathbb{L}$ may be found in http://www.uoregon.edu/~koch/math431/LongLine.pdf, which are notes written by Richard Koch at the University of Oregon. Once we have $\mathbb{W}$, we define $\mathbb{Y}$ to be the union of $\mathbb{W}$, $\{0\}$ and a second copy of $\mathbb{W}$ that we denote $-\mathbb{W}$. The elements of $-\mathbb{W}$ are denoted $-w$ with $w \in \mathbb{W}$. We introduce a total ordering on $\mathbb{Y}$ by requiring that $-w < 0 < w'$ and that $-w < -w' \iff w' < w$ for all $w, w' \in \mathbb{W}$. Then we define $\mathbb{L}$ to be the set $\bigcup_{\ell \in \mathbb{Y}} I_\ell$ where, again, $I_\ell = \{\ (\ell, x)\ \big|\ x \in [0,1)\ \}$ and introduce a topology as above. Then

- $\mathbb{L}$ is totally ordered.
- For each $r, s \in \mathbb{L}$, the interval defined by $(r,s) = \{\ t \in \mathbb{L}\ \big|\ r < t < s\ \}$ is open and homeomorphic to $(0,1)$ in $\mathbb{R}$, with the homeomorphism being order preserving. Every open set in the long line is a union of such open intervals.
- $\mathbb{L}$ is not second countable, because $\{\ \mathcal{O}_\ell\ \big|\ \ell \in \mathbb{Y}\ \}$, with $\mathcal{O}_\ell = \{\ (\ell, x)\ \big|\ x \in (\frac{1}{4}, \frac{3}{4})\ \}$ is an uncountable collection of disjoint nonempty open subsets of $\mathbb{L}$. The ordinary line $\mathbb{R}$ is homeomorphic to the open interval $(0,1)$. But the long line is not homeomorphic to any subset of $\mathbb{R}^n$ because it is not second countable.

An atlas for $\mathbb{L}$ is $\mathcal{A} = \{\ (\mathcal{U}_{r,s}, \varphi_{r,s})\ \big|\ r, s \in \mathbb{L},\ r < s\ \}$ with $\mathcal{U}_{r,s} = (r,s)$ and $\varphi_{r,s}$ being the homeomorphism referred to above.

