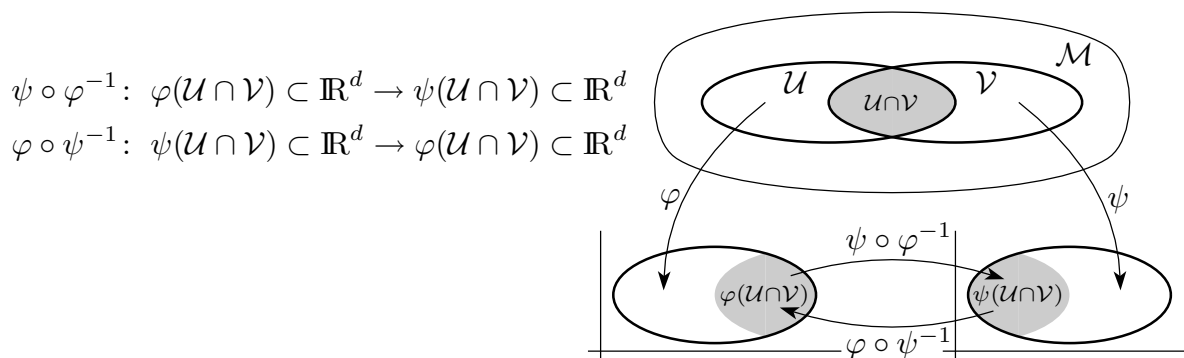


Examples of Manifolds

A manifold is a generalization of a surface. Roughly speaking, a d -dimensional manifold is a set that looks locally like \mathbb{R}^d . It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of \mathbb{R}^d . Here is a precise definition.

Definition 1 We now define what is meant by the statement that \mathcal{M} is a d -dimensional manifold of class C^k (with $1 \leq k \leq \infty$ — we shall deal almost exclusively with manifolds of class C^∞).

- (a) Let \mathcal{M} be a Hausdorff topological space⁽¹⁾. A *coordinate system* (or *chart* or *coordinate patch*) on \mathcal{M} is a pair (\mathcal{U}, φ) with \mathcal{U} a connected open subset of \mathcal{M} and φ a homeomorphism (a 1-1, onto, continuous function with continuous inverse) from \mathcal{U} onto an open subset of \mathbb{R}^d . Think of φ as assigning coordinates to each point of \mathcal{U} . A coordinate system (\mathcal{U}, φ) is called a *cubic coordinate system* if $\varphi(\mathcal{U})$ is an open cube about the origin in \mathbb{R}^d . (That is, if there are numbers $a_1, \dots, a_d, b_1, \dots, b_d > 0$ such that $\varphi(\mathcal{U}) = \{ x \in \mathbb{R}^d \mid -a_i < x_i < b_i, \text{ for all } 1 \leq i \leq d \}$.) If $m \in \mathcal{U}$ and $\varphi(m) = 0$, then the coordinate system is said to be *centred at m* .
- (b) A *locally Euclidean space of dimension d* , is a Hausdorff topological space for which every point has a neighbourhood that is homeomorphic to an open subset of \mathbb{R}^d .
- (c) Two charts (\mathcal{U}, φ) and (\mathcal{V}, ψ) are said to be *compatible* of class C^k if the transition functions



are C^k . That is, all partial derivatives up to order k (for C^∞ , all partial derivatives of all orders) of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ exist and are continuous.

- (d) An *atlas* of class C^k for a locally Euclidean space \mathcal{M} is a family $\mathcal{A} = \{ (\mathcal{U}_i, \varphi_i) \mid i \in \mathcal{I} \}$ of coordinate systems on \mathcal{M} such that $\bigcup_{i \in \mathcal{I}} \mathcal{U}_i = \mathcal{M}$ and such that every pair of charts

⁽¹⁾ If you don't know what this means, substitute "metric space" for "Hausdorff topological space" and read the notes "A Little Point Set Topology".

in \mathcal{A} is compatible of class C^k . The index set \mathcal{I} is completely arbitrary. It could consist of just a single index. It could consist of uncountably many indices. An atlas \mathcal{A} is called *maximal* of class C^k if every chart (U, φ) on \mathcal{M} that is compatible of class C^k with every chart of \mathcal{A} is itself in \mathcal{A} . A maximal atlas of class C^k is also called a *differentiable structure of class C^k* .

- (e) An *d-dimensional manifold of class C^k* is a pair $(\mathcal{M}, \mathcal{A})$ with \mathcal{M} a d -dimensional, second countable, locally Euclidean space and \mathcal{A} a differentiable structure of class C^k . “Second countable” means that \mathcal{M} has a countable base. A *base* is a collection \mathcal{B} of open subsets of \mathcal{M} with the property that every open subset of \mathcal{M} is a union of elements of \mathcal{B} . A metric space that contains a countable dense subset (the metric space is then said to be separable) is automatically second countable. The countable base is the set of all open balls $B_r(x) = \{ y \in \mathcal{M} \mid d(x, y) < r \}$ with centre x in the countable dense subset and radius r a positive rational number.

Problem 1 Let \mathcal{A} be an atlas for the Hausdorff space \mathcal{M} . Prove that there is a unique maximal atlas for \mathcal{M} that contains \mathcal{A} .

Problem 2 Let \mathcal{U} and \mathcal{V} be open subsets of a Hausdorff space \mathcal{M} . Let φ be a homeomorphism from \mathcal{U} to an open subset of \mathbb{R}^n and ψ be a homeomorphism from \mathcal{V} to an open subset of \mathbb{R}^m . Prove that if $\mathcal{U} \cap \mathcal{V}$ is nonempty and

$$\begin{aligned} \psi \circ \varphi^{-1}: \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n &\rightarrow \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m \\ \varphi \circ \psi^{-1}: \psi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^m &\rightarrow \varphi(\mathcal{U} \cap \mathcal{V}) \subset \mathbb{R}^n \end{aligned}$$

are C^1 , then $m = n$.

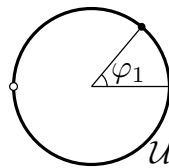
Thanks to Problem 1, it suffices to supply any, not necessarily maximal, atlas for a second countable Hausdorff space to turn it into a manifold. We do exactly that in each of the following examples.

Example 2 (Open Subset of \mathbb{R}^d) \mathbb{R}^d is a metric space and hence is a Hausdorff topological space. It is second countable because the set of all open balls with rational radii and rational centres is a countable base. Let $\mathbb{1}_d$ be the identity map on \mathbb{R}^d . Then $\{(\mathbb{R}^d, \mathbb{1}_d)\}$ is an atlas for \mathbb{R}^d . Indeed, if \mathcal{U} is any nonempty, open subset of \mathbb{R}^d , then $\{(\mathcal{U}, \mathbb{1}_d)\}$ is an atlas for \mathcal{U} . So every open subset of \mathbb{R}^d is naturally a C^∞ manifold.

Example 3 (The Circle) The circle $S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ is a manifold of dimension one when equipped with, for example, the atlas $\mathcal{A} = \{(\mathcal{U}_1, \varphi_1), (\mathcal{U}_2, \varphi_2)\}$ where

$$\mathcal{U}_1 = S^1 \setminus \{(-1, 0)\} \quad \varphi_1(x, y) = \arctan \frac{y}{x} \text{ with } -\pi < \varphi_1(x, y) < \pi$$

$$\mathcal{U}_2 = S^1 \setminus \{(1, 0)\} \quad \varphi_2(x, y) = \arctan \frac{y}{x} \text{ with } 0 < \varphi_2(x, y) < 2\pi$$



My use of $\arctan \frac{y}{x}$ here is pretty sloppy. To define φ_1 carefully, we can say that $\varphi_1(x, y)$ is the unique $-\pi < \theta < \pi$ such that $(x, y) = (\cos \theta, \sin \theta)$. To verify that these two charts are compatible, we first determine the domain intersection $\mathcal{U}_1 \cap \mathcal{U}_2 = S^1 \setminus \{(-1, 0), (1, 0)\}$ and then the ranges $\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) = (-\pi, 0) \cup (0, \pi)$ and $\varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2) = (0, \pi) \cup (\pi, 2\pi)$ and finally, we check that

$$\varphi_2 \circ \varphi_1^{-1}(\theta) = \begin{cases} \theta & \text{if } 0 < \theta < \pi \\ \theta + 2\pi & \text{if } -\pi < \theta < 0 \end{cases} \quad \varphi_1 \circ \varphi_2^{-1}(\theta) = \begin{cases} \theta & \text{if } 0 < \theta < \pi \\ \theta - 2\pi & \text{if } \pi < \theta < 2\pi \end{cases}$$

are indeed C^∞ .

Example 4 (The d -Sphere) The d -sphere

$$S^d = \{ \mathbf{x} = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1^2 + \dots + x_{d+1}^2 = 1 \}$$

is a manifold of dimension d when equipped with the atlas

$$\mathcal{A}_1 = \{ (\mathcal{U}_i, \varphi_i), (\mathcal{V}_i, \psi_i) \mid 1 \leq i \leq d+1 \}$$

where, for each $1 \leq i \leq d+1$,

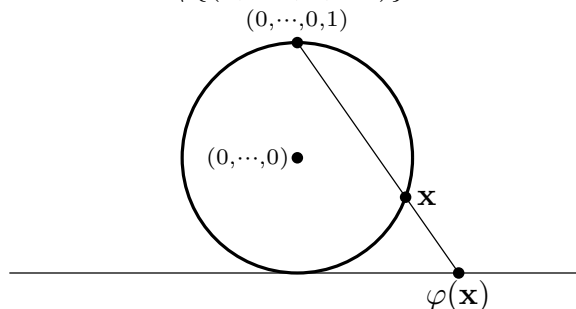
$$\mathcal{U}_i = \{ (x_1, \dots, x_{d+1}) \in S^d \mid x_i > 0 \} \quad \varphi_i(x_1, \dots, x_{d+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1})$$

$$\mathcal{V}_i = \{ (x_1, \dots, x_{d+1}) \in S^d \mid x_i < 0 \} \quad \psi_i(x_1, \dots, x_{d+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{d+1})$$

So both φ_i and ψ_i just discard the coordinate x_i . They project onto \mathbb{R}^d , viewed as the hyperplane $x_i = 0$. Another possible atlas, compatible with \mathcal{A}_1 , is $\mathcal{A}_2 = \{ (\mathcal{U}, \varphi), (\mathcal{V}, \psi) \}$ where the domains $\mathcal{U} = S^d \setminus \{(0, \dots, 0, 1)\}$ and $\mathcal{V} = S^d \setminus \{(0, \dots, 0, -1)\}$ and

$$\varphi(x_1, \dots, x_{d+1}) = \left(\frac{2x_1}{1-x_{d+1}}, \dots, \frac{2x_d}{1-x_{d+1}} \right)$$

$$\psi(x_1, \dots, x_{d+1}) = \left(\frac{2x_1}{1+x_{d+1}}, \dots, \frac{2x_d}{1+x_{d+1}} \right)$$



are the stereographic projections from the north and south poles, respectively. Both φ and ψ have range \mathbb{R}^d . So we can think of S^d as \mathbb{R}^d plus an additional single “point at infinity”.

Problem 3 In this problem we use the notation of Example 4.

(a) Prove that \mathcal{A}_1 is an atlas for S^d .

(b) Prove that \mathcal{A}_2 is an atlas for S^d .

Example 5 (Surfaces) Any smooth n -dimensional surface in \mathbb{R}^{n+m} is an n -dimensional manifold. Roughly speaking, a subset of \mathbb{R}^{n+m} is an n -dimensional surface if, locally, m of the $m+n$ coordinates of points on the surface are determined by the other n coordinates in a C^∞ way. For example, the unit circle S^1 is a one dimensional surface in \mathbb{R}^2 . Near $(0, 1)$ a point $(x, y) \in \mathbb{R}^2$ is on S^1 if and only if $y = \sqrt{1 - x^2}$, and near $(-1, 0)$, (x, y) is on S^1 if and only if $x = -\sqrt{1 - y^2}$.

The precise definition is that \mathcal{M} is an n -dimensional surface in \mathbb{R}^{n+m} if \mathcal{M} is a subset of \mathbb{R}^{n+m} with the property that for each $\mathbf{z} = (z_1, \dots, z_{n+m}) \in \mathcal{M}$, there are

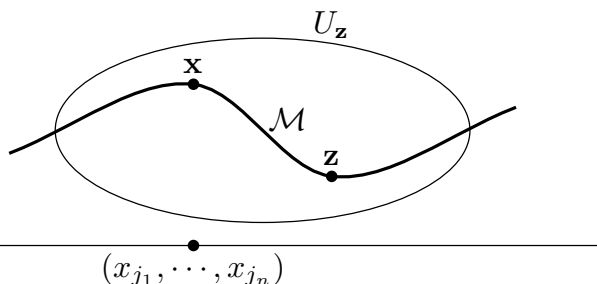
- a neighbourhood $U_{\mathbf{z}}$ of \mathbf{z} in \mathbb{R}^{n+m}
- n integers $1 \leq j_1 < j_2 < \dots < j_n \leq n + m$
- and m C^∞ functions $f_k(x_{j_1}, \dots, x_{j_n})$, $k \in \{1, \dots, n + m\} \setminus \{j_1, \dots, j_n\}$

such that the point $\mathbf{x} = (x_1, \dots, x_{n+m}) \in U_{\mathbf{z}}$ is in \mathcal{M} if and only if $x_k = f_k(x_{j_1}, \dots, x_{j_n})$ for all $k \in \{1, \dots, n + m\} \setminus \{j_1, \dots, j_n\}$. That is, we may express the part of \mathcal{M} that is near \mathbf{z} as

$$\begin{aligned} x_{i_1} &= f_{i_1}(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \\ x_{i_2} &= f_{i_2}(x_{j_1}, x_{j_2}, \dots, x_{j_n}) \\ &\vdots \end{aligned}$$

$$x_{i_m} = f_{i_m}(x_{j_1}, x_{j_2}, \dots, x_{j_n})$$

$$\text{where } \{i_1, \dots, i_m\} = \{1, \dots, n + m\} \setminus \{j_1, \dots, j_n\}$$



for some C^∞ functions f_1, \dots, f_m . We may use $x_{j_1}, x_{j_2}, \dots, x_{j_n}$ as coordinates for \mathcal{M} in $\mathcal{M} \cap U_{\mathbf{z}}$. Of course, an atlas is $\mathcal{A} = \{ (U_{\mathbf{z}} \cap \mathcal{M}, \varphi_{\mathbf{z}}) \mid \mathbf{z} \in \mathcal{M} \}$, with $\varphi_{\mathbf{z}}(\mathbf{x}) = (x_{j_1}, \dots, x_{j_n})$.

Equivalently, \mathcal{M} is an n -dimensional surface in \mathbb{R}^{n+m} , if, for each $\mathbf{z} \in \mathcal{M}$, there are

- a neighbourhood $U_{\mathbf{z}}$ of \mathbf{z} in \mathbb{R}^{n+m}
- and m C^∞ functions $g_k : U_{\mathbf{z}} \rightarrow \mathbb{R}$, with the vectors $\{ \nabla g_k(\mathbf{z}) \mid 1 \leq k \leq m \}$ linearly independent

such that the point $\mathbf{x} \in U_{\mathbf{z}}$ is in \mathcal{M} if and only if $g_k(\mathbf{x}) = 0$ for all $1 \leq k \leq m$. To get from the implicit equations for \mathcal{M} given by the g_k 's to the explicit equations for \mathcal{M} given by the f_k 's one need only invoke (possibly after renumbering the components of \mathbf{x}) the

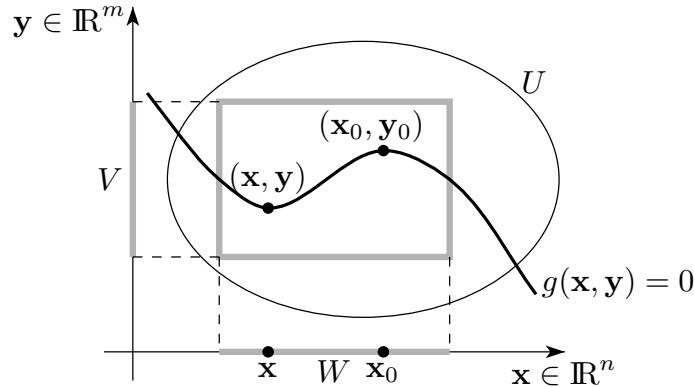
Implicit Function Theorem

Let $m, n \in \mathbb{N}$ and let $U \subset \mathbb{R}^{n+m}$ be an open set. Let $\mathbf{g} : U \rightarrow \mathbb{R}^m$ be C^∞ with $\mathbf{g}(\mathbf{x}_0, \mathbf{y}_0) = 0$ for some $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{y}_0 \in \mathbb{R}^m$ with $(\mathbf{x}_0, \mathbf{y}_0) \in U$. Assume that

$\det \left[\frac{\partial g_i}{\partial y_j}(\mathbf{x}_0, \mathbf{y}_0) \right]_{1 \leq i, j \leq m} \neq 0$. Then there exist open sets $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ with $\mathbf{x}_0 \in W$ and $\mathbf{y}_0 \in V$ such that

for each $\mathbf{x} \in W$, there is a unique $\mathbf{y} \in V$ with $\mathbf{g}(\mathbf{x}, \mathbf{y}) = 0$.

If the \mathbf{y} above is denoted $\mathbf{f}(\mathbf{x})$, then $\mathbf{f} : W \rightarrow \mathbb{R}^m$ is C^∞ , $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$ and $\mathbf{g}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = 0$ for all $\mathbf{x} \in W$.



The d -sphere S^d is the d -dimensional surface in \mathbb{R}^{d+1} given implicitly by the equation $g(x_1, \dots, x_{d+1}) = x_1^2 + \dots + x_{d+1}^2 - 1 = 0$. In a neighbourhood of the north pole (for example, the northern hemisphere), S^d is given explicitly by the equation $x_{d+1} = \sqrt{x_1^2 + \dots + x_d^2}$. If you think of the set of all 3×3 real matrices as \mathbb{R}^9 (because a 3×3 matrix has 9 matrix elements) then

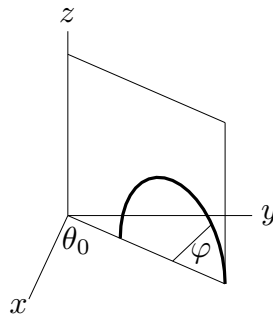
$$SO(3) = \{ 3 \times 3 \text{ real matrices } R \mid R^t R = \mathbb{1}, \det R = 1 \}$$

is a 3-dimensional surface in \mathbb{R}^9 . We shall look at it more closely in Example 7, below. $SO(3)$ is the group of all rotations about the origin in \mathbb{R}^3 and is also the set of all orientations of a rigid body with one point held fixed.

Example 6 (A Torus) The torus T^2 is the two dimensional surface

$$T^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - 1)^2 + z^2 = \frac{1}{4} \}$$

in \mathbb{R}^3 . In cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, the equation of the torus is $(r - 1)^2 + z^2 = \frac{1}{4}$. Fix any θ , say θ_0 . Recall that the set of all points in



\mathbb{R}^3 that have $\theta = \theta_0$ is like one page in an open book. It is a half-plane that starts at the z axis. The intersection of the torus with that half plane is a circle of radius $\frac{1}{2}$ centred on $r = 1, z = 0$. As φ runs from 0 to 2π , the point $r = 1 + \frac{1}{2} \cos \varphi, z = \frac{1}{2} \sin \varphi, \theta = \theta_0$ runs over that circle. If we now run θ from 0 to 2π , the circle on the page sweeps out the whole torus. So, as φ runs from 0 to 2π and θ runs from 0 to 2π , the point $(x, y, z) = ((1 + \frac{1}{2} \cos \varphi) \cos \theta, (1 + \frac{1}{2} \cos \varphi) \sin \theta, \frac{1}{2} \sin \varphi)$ runs over the whole torus. So we may build coordinate patches for T^2 using θ and φ (with ranges $(0, 2\pi)$ or $(-\pi, \pi)$) as coordinates.

Example 7 ($O(3), SO(3)$) As a special case of Example 5 we have the groups

$$SO(3) = \{ 3 \times 3 \text{ real matrices } R \mid R^t R = \mathbb{1}_3, \det R = 1 \}$$

$$O(3) = \{ 3 \times 3 \text{ real matrices } R \mid R^t R = \mathbb{1}_3 \}$$

of rotations and rotations/reflections in \mathbb{R}^3 . (Rotations and reflections are the angle and length preserving linear maps. In classical mechanics, $SO(3)$ is the set of all possible configurations of rigid body with one point held fixed.) We can identify the set of all 3×3 real matrices with \mathbb{R}^9 , because a 3×3 matrix has 9 matrix elements. The restriction that

$$R = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \in O(3)$$

is given implicitly by the following six equations.

$$\begin{aligned} (R^t R)_{1,1} &= a_1^2 + a_2^2 + a_3^2 = 1 && \text{i.e. } |\mathbf{a}| = 1 \\ (R^t R)_{2,2} &= b_1^2 + b_2^2 + b_3^2 = 1 && \text{i.e. } |\mathbf{b}| = 1 \\ (R^t R)_{3,3} &= c_1^2 + c_2^2 + c_3^2 = 1 && \text{i.e. } |\mathbf{c}| = 1 \\ (R^t R)_{1,2} &= (R^t R)_{2,1} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 0 && \text{i.e. } \mathbf{a} \perp \mathbf{b} \\ (R^t R)_{1,3} &= (R^t R)_{3,1} = a_1 c_1 + a_2 c_2 + a_3 c_3 = 0 && \text{i.e. } \mathbf{a} \perp \mathbf{c} \\ (R^t R)_{2,3} &= (R^t R)_{3,2} = b_1 c_1 + b_2 c_2 + b_3 c_3 = 0 && \text{i.e. } \mathbf{b} \perp \mathbf{c} \end{aligned} \tag{1}$$

We can verify the independence conditions of Example 5 (that the gradients of the left hand sides are independent) directly. See Problems 5 and 6, below. Or we can argue geometrically. In a neighbourhood of any fixed element, \tilde{R} , of $SO(3)$, we may use two of the three \mathbf{a} -components as coordinates. (In fact we may use any two \mathbf{a} -coordinates whose magnitude at \tilde{R} is not one.) Once two components of \mathbf{a} have been chosen, the third \mathbf{a} -component is determined up to a sign by the requirement that $|\mathbf{a}| = 1$. The sign is chosen so as to remain in the neighbourhood. Once \mathbf{a} has been chosen, the set $\{ \mathbf{b} \in \mathbb{R}^3 \mid \mathbf{b} \perp \mathbf{a} \}$ is a plane through the origin so that $\{ \mathbf{b} \in \mathbb{R}^3 \mid \mathbf{b} \perp \mathbf{a}, |\mathbf{b}| = 1 \}$ is the intersection of

that plane with the unit sphere. So \mathbf{b} lies on a great circle of the unit sphere. Thus \mathbf{b} is determined up to a single rotation angle by the requirements that $\mathbf{b} \perp \mathbf{a}$ and $|\mathbf{b}| = 1$. That rotation angle is the third coordinate. Once \mathbf{a} and \mathbf{b} have been chosen, the set $\{ \mathbf{c} \in \mathbb{R}^3 \mid \mathbf{c} \perp \mathbf{a}, \mathbf{c} \perp \mathbf{b} \}$ is a line through the origin. So \mathbf{c} is determined up to a sign by the requirements that $\mathbf{c} \perp \mathbf{a}, \mathbf{b}$ and $|\mathbf{c}| = 1$. Again, the sign is chosen so as to remain in the neighbourhood. So $O(3)$ is a manifold of dimension 3. Any element of $O(3)$ automatically obeys

$$(\det R)^2 = \det R^t R = \det \mathbb{1}_3 = 1 \implies \det R = \pm 1$$

So $SO(3)$ is just one of the two connected components of $O(3)$. It is an important example of a Lie group, which is, by definition, a C^∞ manifold that is also a group with the operations of multiplication and taking inverses continuous.

Problem 4 Let $R \in O(3)$.

- Prove that if λ is an eigenvalue of R , then $|\lambda| = 1$ and $\bar{\lambda}$ is an eigenvalue of R .
- Prove that at least one eigenvalue of R is either $+1$ or -1 .
- Prove that the columns of R are mutually perpendicular and are each of unit length.
- Prove that R is either a rotation, a reflection or a composition of a rotation and a reflection.

Problem 5 Denote by g_1, \dots, g_6 the left hand sides of (1). Prove that the gradients of g_1, \dots, g_6 , evaluated at any $R \in O(3)$, are linearly independent.

Problem 6 Use the implicit function theorem to prove that for each $1 \leq i, j \leq 3$, the (i, j) matrix element, a_{ij} , of matrices $R = [a_{ij}]_{1 \leq i, j \leq 3}$ in a neighbourhood of $\mathbb{1}$ in $SO(3)$, is a C^∞ function of the matrix elements a_{21} , a_{31} and a_{32} .

Example 8 (More Tori) Define an equivalence relation on \mathbb{R}^d by

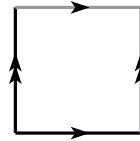
$$x \sim y \iff x - y \in \mathbb{Z}^d$$

In this example, when $x \sim y$ we want to think of x and y as two different names for the same object. The set of all possible names for the object whose name is also x is $[x] = \{ y \in \mathbb{R}^d \mid y \sim x \}$ and is called the equivalence class of $x \in \mathbb{R}^d$. The set of equivalence classes is denoted $\mathbb{R}^d / \mathbb{Z}^d = \{ [x] \mid x \in \mathbb{R}^d \}$. Each equivalence class $[x]$ contains exactly one representative $\tilde{x} \in [x]$ obeying $0 \leq \tilde{x}_j < 1$ for each $1 \leq j \leq d$. So we can also think of $\mathbb{R}^d / \mathbb{Z}^d$ as being

$$\{ x \in \mathbb{R}^d \mid 0 \leq x_j < 1 \text{ for all } 1 \leq j \leq d \}$$

But then we should also identify, for each $1 \leq j \leq d$, the edges

$$\begin{aligned} & \{ x \in \mathbb{R}^d \mid x_j = 1, 0 \leq x_i \leq 1 \forall i \neq j \} \\ \text{and } & \{ x \in \mathbb{R}^d \mid x_j = 0, 0 \leq x_i \leq 1 \forall i \neq j \} \end{aligned}$$



We can turn the set $\mathbb{R}^d/\mathbb{Z}^d$, which is also called a torus, into a metric space by imposing the metric

$$\rho([x], [y]) = \min \{ |\tilde{x} - \tilde{y}| \mid \tilde{x} \in [x], \tilde{y} \in [y] \}$$

So we only need an atlas to turn the torus into a manifold. If \mathcal{U} is any open subset of \mathbb{R}^d with the property that no two points of \mathcal{U} are equivalent (any open ball of radius at most $\frac{1}{2}$ has this property), then $[\mathcal{U}] = \{ [x] \mid x \in \mathcal{U} \}$ is an open subset of $\mathbb{R}^d/\mathbb{Z}^d$ and each element of $[\mathcal{U}]$ contains a unique representative $\tilde{x} \in [x]$ that is in \mathcal{U} . Define

$$\begin{aligned} \Phi_{\mathcal{U}} : [\mathcal{U}] &\rightarrow \mathbb{R}^d \\ [x] &\mapsto \tilde{x} \text{ with } \tilde{x} \in [x], \tilde{x} \in \mathcal{U} \end{aligned}$$

Then $\{[\mathcal{U}], \Phi_{\mathcal{U}}\}$ is a chart and the set of all such charts is an atlas.

Example 9 (The Cartesian Product) If \mathcal{M} is a manifold of dimension m with atlas \mathcal{A} and \mathcal{N} is a manifold of dimension n with atlas \mathcal{B} then

$$\mathcal{M} \times \mathcal{N} = \{ (x, y) \mid x \in \mathcal{M}, y \in \mathcal{N} \}$$

is an $(m+n)$ -dimensional manifold with atlas

$$\{ (U \times V, \varphi \oplus \psi) \mid (U, \varphi) \in \mathcal{A}, (V, \psi) \in \mathcal{B} \} \quad \text{where } \varphi \oplus \psi((x, y)) = (\varphi(x), \psi(y))$$

For example, $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, $S^1 \times \mathbb{R}$ is a cylinder, $S^1 \times S^1$ is a torus and the configuration space of a rigid body is $\mathbb{R}^3 \times SO(3)$ (with the \mathbb{R}^3 components giving the location of the centre of mass of the body and the $SO(3)$ components giving the orientation).

Example 10 (The Möbius Strip) We are now going to turn the set

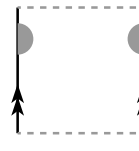
$$\mathcal{M} = [0, 1) \times (-1, 1) = \{ (s, t) \mid 0 \leq s < 1, -1 < t < 1 \}$$

into two very different manifolds by assigning two different, incompatible, atlases. Both atlases will contain two charts with

$$\mathcal{U}_1 = \left(\frac{1}{8}, \frac{7}{8}\right) \times (-1, 1) \quad \mathcal{U}_2 = \left[0, \frac{1}{4}\right) \times (-1, 1) \cup \left(\frac{3}{4}, 1\right) \times (-1, 1)$$

The first atlas attaches each point $(0, t)$ on the left hand edge to the point $(1, t)$ on the right hand edge by using the coordinate functions

$$(x, y) = \psi_1(s, t) = (s, t)$$

$$(x, y) = \psi_2(s, t) = \begin{cases} (s, t) & \text{if } 0 \leq s < \frac{1}{4} \\ (s - 1, t) & \text{if } \frac{3}{4} < s < 1 \end{cases}$$


The range of ψ_2 is

$$\begin{aligned} \psi_2\left(\left[0, \frac{1}{4}\right] \times (-1, 1)\right) \cup \psi_2\left(\left(\frac{3}{4}, 1\right) \times (-1, 1)\right) &= \left[0, \frac{1}{4}\right] \times (-1, 1) \cup \left(-\frac{1}{4}, 0\right) \times (-1, 1) \\ &= \left(-\frac{1}{4}, \frac{1}{4}\right) \times (-1, 1) \end{aligned}$$

The inverse map for ψ_2 is

$$(s, t) = \psi_2^{-1}(x, y) = \begin{cases} (x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ (x + 1, y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

The inverse image under ψ_2 of the disk $x^2 + (y - \frac{1}{2})^2 < \frac{1}{16}$ (denote it $B_{\frac{1}{4}}(0, \frac{1}{2})$) is

$$\begin{aligned} \psi_2^{-1}\left(B_{\frac{1}{4}}\left(0, \frac{1}{2}\right) \cap \{x \geq 0\}\right) \cup \psi_2^{-1}\left(B_{\frac{1}{4}}\left(0, \frac{1}{2}\right) \cap \{x < 0\}\right) \\ = B_{\frac{1}{4}}\left(0, \frac{1}{2}\right) \cap \{x \geq 0\} \cup \left\{ (x + 1, y) \mid (x, y) \in B_{\frac{1}{4}}\left(0, \frac{1}{2}\right), x < 0 \right\} \end{aligned}$$

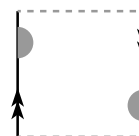
That is the union of the two shaded half disks displayed in the figure above. The union is connected in the manifold with atlas $\{\{\mathcal{U}_1, \psi_1\}, \{\mathcal{U}_1, \psi_2\}\}$. This manifold may be constructed from a strip of paper by gluing the left and right hand edges together. To complete the definition of this manifold, it suffices to provide it with a metric and then verify that $\{(\mathcal{U}_1, \psi_1), (\mathcal{U}_1, \psi_2)\}$ really is an atlas and, in particular, that ψ_2 and its inverse are continuous. The metric (similar to the metric of Example 8)

$$\rho_\psi((s, t), (s', t')) = \min \left\{ |(s - s', t - t')|, |(s - s' + 1, t - t')|, |(s - s' - 1, t - t')| \right\}$$

works.

The second atlas attaches each point $(0, t)$ on the left hand edge to the point $(1, -t)$ on the right hand edge by using the coordinate functions

$$(x, y) = \varphi_1(s, t) = (s, t)$$

$$(x, y) = \varphi_2(s, t) = \begin{cases} (s, t) & \text{if } 0 \leq s < \frac{1}{4} \\ (s - 1, -t) & \text{if } \frac{3}{4} < s < 1 \end{cases}$$


The range of φ_2 is $(-\frac{1}{4}, \frac{1}{4}) \times (-1, 1)$, the same as the range of ψ_2 . The inverse map for φ_2 is

$$(s, t) = \varphi_2^{-1}(x, y) = \begin{cases} (x, y) & \text{if } 0 \leq x < \frac{1}{4} \\ (x + 1, -y) & \text{if } -\frac{1}{4} < x < 0 \end{cases}$$

The union of the two shaded half disks in the figure above is the inverse image under φ_2 of the disk $x^2 + (y - \frac{1}{2})^2 < \frac{1}{16}$. That union is connected in the manifold with atlas $\{\{\mathcal{U}_1, \varphi_1\}, \{\mathcal{U}_2, \varphi_2\}\}$. This manifold may be constructed from a strip of paper by gluing the left and right hand edges together, after putting a half twist in the strip. It is called a Möbius strip. It has metric

$$\rho_\psi((s, t), (s', t')) = \min \{ |(s - s', t - t')|, |(s - s' + 1, t + t')|, |(s - s' - 1, t + t')| \}$$

Problem 7 Prove that the two charts $(\mathcal{U}_2, \varphi_2)$ and (\mathcal{U}_2, ψ_2) of Example 10 are not compatible.

Example 11 (Projective n -space, \mathbb{P}^n) The projective n -space, \mathbb{P}^n , is the set of all lines through the origin in \mathbb{R}^{n+1} . If $\vec{x} \in \mathbb{R}^{n+1}$ is nonzero, then there is a unique line $L_{\vec{x}}$ through the origin in \mathbb{R}^{n+1} that contains \vec{x} . Namely $L_{\vec{x}} = \{ \lambda \vec{x} \mid \lambda \in \mathbb{R} \}$. If $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero, then $L_{\vec{x}} = L_{\vec{y}}$ if and only if there is a $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\vec{y} = \lambda \vec{x}$. One choice of atlas for \mathbb{P}^n is $\mathcal{A} = \{ (U_i, \varphi_i) \mid 1 \leq i \leq n+1 \}$ with

$$U_i = \{ L_{\vec{x}} \mid \vec{x} \in \mathbb{R}^{n+1}, x_i \neq 0 \} \quad \varphi(L_{\vec{x}}) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right) \in \mathbb{R}^n$$

Observe that if φ_i is well-defined, because if $\vec{x}, \vec{y} \in \mathbb{R}^{n+1}$ are both nonzero and $L_{\vec{x}} = L_{\vec{y}}$, then, for each $1 \leq i \leq n+1$, either both x_i and y_i are zero or both x_i and y_i are nonzero and in the latter case

$$\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right) = \left(\frac{y_1}{y_i}, \dots, \frac{y_{i-1}}{y_i}, \frac{y_{i+1}}{y_i}, \dots, \frac{y_{n+1}}{y_i} \right)$$

Each line through the origin in \mathbb{R}^{n+1} intersects the unit sphere $S^n = \{ \vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1 \}$ in exactly two points and the two points are antipodal (i.e. \vec{x} and $-\vec{x}$). So you can think of \mathbb{P}^n as S^n but with antipodal points identified:

$$\mathbb{P}^{n+1} = \{ \{ \vec{x}, -\vec{x} \} \mid \vec{x} \in S^n \}$$

Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is not horizontal (i.e. with $x_{n+1} \neq 0$) intersects the northern hemisphere $\{ \vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1, x_{n+1} \geq 0 \}$ in exactly one point. Each line $L_{\vec{x}} \in \mathbb{P}^n$ that is horizontal (i.e. with $x_{n+1} = 0$) intersects the northern hemisphere in exactly two points and the two points are antipodal. By ignoring x_{n+1} , you can think of the northern hemisphere as the closed unit disk $\{ \mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1 \}$ in \mathbb{R}^n . So you can think of \mathbb{P}^n as the closed unit ball in \mathbb{R}^n but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified.

In the case of three dimensions, you can also think of $SO(3)$ as being the closed unit disk $\{ \mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| \leq 1 \} \subset \mathbb{R}^3$ but with antipodal points on the boundary $|\mathbf{x}| = 1$ identified. This is because, geometrically, each element of $SO(3)$ is a matrix which implements a rotation by some angle about some axis through the origin in \mathbb{R}^3 . We can associate each $\omega\hat{\Omega} \in \mathbb{R}^3$, where $\hat{\Omega}$ is a unit vector and $\omega \in \mathbb{R}$, with the rotation by an angle $\pi\omega$ about the axis $\hat{\Omega}$. But then any two ω 's that differ by an even integer give the same rotation. So the set of all rotations is associated with $\{ \omega\hat{\Omega} \mid |\omega| \leq 1, \hat{\Omega} \in \mathbb{R}^3, |\hat{\Omega}| = 1 \}$ but with $1\hat{\Omega}$ and $-1\hat{\Omega}$ identified. Thus $SO(3)$ and \mathbb{P}^3 are diffeomorphic, where

Definition 12

- (a) A function f from a manifold \mathcal{M} to a manifold \mathcal{N} (it is traditional to omit the atlas from the notation) is said to be C^∞ at $m \in \mathcal{M}$ if there exists a chart (\mathcal{U}, φ) for \mathcal{M} and a chart (\mathcal{V}, ψ) for \mathcal{N} such that $m \in \mathcal{U}$, $f(m) \in \mathcal{V}$ and $\psi \circ f \circ \varphi^{-1}$ is C^∞ at $\varphi(m)$.
- (b) Two manifolds \mathcal{M} and \mathcal{N} are *diffeomorphic* if there exists a function $f : \mathcal{M} \rightarrow \mathcal{N}$ that is 1-1 and onto with f and f^{-1} C^∞ everywhere. Then you should think of \mathcal{M} and \mathcal{N} as the same manifold with m and $f(m)$ being two different names for the same point, for each $m \in \mathcal{M}$.

Problem 8 Let \mathcal{M} and \mathcal{N} be manifolds. Prove that $f : \mathcal{M} \rightarrow \mathcal{N}$ is C^∞ at $m \in \mathcal{M}$ if and only if $\psi \circ f \circ \phi^{-1}$ is C^∞ at $\phi(m)$ for every chart (\mathcal{U}, ϕ) for \mathcal{M} with $m \in \mathcal{U}$ and every chart (\mathcal{V}, ψ) for \mathcal{N} with $f(m) \in \mathcal{V}$.

Problem 9 Prove that \mathbb{R}^n is diffeomorphic to $\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < 1 \}$.

Problem 10 Prove that \mathbb{R}^n is not diffeomorphic to S^n .

Problem 11 Outline an argument to prove that the disk $\{ \mathbf{x} \in \mathbb{R}^2 \mid x^2 + y^2 < 2 \}$ is not diffeomorphic to the annulus $\{ \mathbf{x} \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2 \}$.

Problem 12 In this problem $G = SO(3)$.

- a) Fix any $a \in G$. Denote by $I = \{ (i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq 3, 1 \leq j \leq 3 \}$ the set of indices for the matrix elements of the matrices in G . Prove that there exist $\alpha, \beta, \gamma \in I$ such that every matrix element g_δ , $\delta \in I$ is a C^∞ function of $g_\alpha, g_\beta, g_\gamma$ for matrices $g \in G$ in a neighbourhood of a .
- b) Prove that a curve $q : (c, d) \rightarrow G$ is C^∞ if and only if every matrix element $q(t)_{i,j}$ is C^∞ .
- c) Prove that matrix multiplication $(a, b) \mapsto ab$ is a C^∞ function from $G \times G$ to G .
- d) Prove that the inverse function $a \mapsto a^{-1}$ is a C^∞ function from G to G .

Example 13 You might think that the Hausdorff requirement that we included in the definition of a manifold is superfluous – that it is a consequence of the requirement that every point has a neighbourhood homeomorphic to an open subset of \mathbb{R}^d . Here is an example that shows otherwise. It satisfies all of the requirements of a manifold except one — it is not Hausdorff.

To start, we just define the set

$$M = (0, 1) \cup \{ (x, \text{red}) \mid 1 \leq x < 2 \} \cup \{ (y, \text{yellow}) \mid 1 \leq y < 2 \}$$

(so that M contains two distinct copies of the interval $[1, 2)$ together with one copy of the interval $(0, 1)$).

Next, we endow M with a topology. We give the subset

$$M_r = (0, 1) \cup \{ (x, \text{red}) \mid 1 \leq x < 2 \}$$

the usual topology of the real interval $(0, 2)$. We also give the subset

$$M_y = (0, 1) \cup \{ (x, \text{yellow}) \mid 1 \leq x < 2 \}$$

the usual topology of the real interval $(0, 2)$. Then we define a subset $S \subset M$ to be open if and only if $S \cap M_r$ and $S \cap M_y$ are open. That is, $S \subset M$ is open if and only if $\{ x \in (0, 1) \mid x \in S \} \cup \{ x \in [1, 2) \mid (x, \text{red}) \in S \}$ and $\{ x \in (0, 1) \mid x \in S \} \cup \{ x \in [1, 2) \mid (x, \text{yellow}) \in S \}$ are both open subsets of $(0, 2)$. This topology is not Hausdorff, because any two open sets U_1 and U_2 with $(1, \text{red}) \in U_1$ and $(1, \text{yellow}) \in U_2$ both necessarily contain $(1 - \varepsilon, 1)$ for some $\varepsilon > 0$ and hence necessarily intersect.

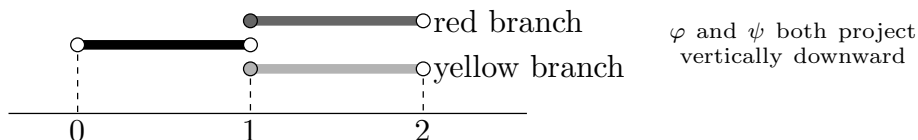
We may nonetheless give M an atlas consisting of the two charts (U, φ) and (V, ψ) where

$$U = M_r = (0, 1) \cup \{ (x, \text{red}) \mid 1 \leq x < 2 \}$$

$$\varphi(x) = x \text{ for all } x \in (0, 1), \quad \varphi((x, \text{red})) = x \text{ for all } x \in [1, 2)$$

$$V = M_y = (0, 1) \cup \{ (y, \text{yellow}) \mid 1 \leq y < 2 \}$$

$$\psi(y) = y \text{ for all } y \in (0, 1), \quad \psi((y, \text{yellow})) = y \text{ for all } y \in [1, 2)$$



These two charts are both homeomorphisms onto $(0, 2)$ and are compatible since

$$U \cap V = (0, 1) \quad \varphi \circ \psi^{-1}(y) = y \text{ and } \psi \circ \varphi^{-1}(x) = x \text{ for all } x, y \in (0, 1)$$

Example 14 Here is an example that satisfies all of the requirements of a manifold except for second countability. It is called the long line. By way of motivation we reformulate the definition of the real line \mathbb{R} as a manifold (but we'll call it $\tilde{\mathbb{R}}$ now), pretending that we do not already know what \mathbb{R} is but that we do know what the finite intervals $[0, 1)$ and (a, b) are. Define, for each integer $\ell \in \mathbb{Z}$, the set of pairs

$$I_\ell = \{ (\ell, x) \mid x \in [0, 1) \}$$

As a set, we define $\tilde{\mathbb{R}}$ to be $\bigcup_{\ell \in \mathbb{Z}} I_\ell$. (Of course, I am thinking of (ℓ, x) as another name for $\ell + x$.) That is, as a set, $\tilde{\mathbb{R}}$ is the union of countably many copies of the half open interval $[0, 1)$. We may define an ordering on $\tilde{\mathbb{R}}$ by

$$(\ell, x) < (\ell', x') \iff \ell < \ell' \text{ or } \ell = \ell', x < x'$$

Next we introduce a topology on $\tilde{\mathbb{R}}$ by defining an open interval to be a subset of $\tilde{\mathbb{R}}$ of the form $(r, s) = \{ (\ell, x) \in \tilde{\mathbb{R}} \mid r < (\ell, x) < s \}$ for some $r, s \in \tilde{\mathbb{R}}$ and defining a subset of $\tilde{\mathbb{R}}$ to be open if it is a union of open intervals. Finally, we introduce the atlas $\mathcal{A} = \{ (\mathcal{U}_{r,s}, \varphi_{r,s}) \mid r, s \in \tilde{\mathbb{R}}, r < s \}$ with

$$\begin{aligned} \mathcal{U}_{r,s} &= (r, s) \\ \varphi_{r,s}((\ell, x)) &= \ell + x \end{aligned}$$

We now define the long line \mathbb{L} by repeating the above construction, but with the integers \mathbb{Z} replaced by another set that we'll denote \mathbb{Y} . We start by replacing the natural numbers \mathbb{N} with a set, \mathbb{W} , called the first uncountable ordinal. It is characterized by the conditions that

- \mathbb{W} is totally ordered. This means that \mathbb{W} is equipped with a binary relation \leq (i.e. a subset of $\mathbb{W} \times \mathbb{W}$) such that the following statements hold for all $a, b, c \in \mathbb{W}$.
 - If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).
 - If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).
 - Either $a \leq b$ or $b \leq a$ (totality).
- \mathbb{W} is a well-ordered set. This means that every nonempty subset of \mathbb{W} has a least element.
- \mathbb{W} is not countable.
- Whenever $a \in \mathbb{W}$ then $\{ i \in \mathbb{W} \mid i \leq a \}$ is countable.

A proof of the existence of the first uncountable ordinal, as well as more details on the construction of \mathbb{L} may be found in <http://www.uoregon.edu/~koch/math431/LongLine.pdf>, which are notes written by Richard Koch at the University of Oregon. Once we have \mathbb{W} , we define \mathbb{Y} to be the union of \mathbb{W} , $\{0\}$ and a second copy of \mathbb{W} that we denote $-\mathbb{W}$. The elements of $-\mathbb{W}$ are denoted $-w$ with $w \in \mathbb{W}$. We introduce a total ordering on \mathbb{Y} by

requiring that $-w < 0 < w'$ and that $-w < -w' \iff w' < w$ for all $w, w' \in \mathbb{W}$. Then we define \mathbb{L} to be the set $\bigcup_{\ell \in \mathbb{Y}} I_\ell$ where, again, $I_\ell = \{ (\ell, x) \mid x \in [0, 1) \}$ and introduce a topology as above. Then

- \mathbb{L} is totally ordered.
- For each $r, s \in \mathbb{L}$, the interval defined by $(r, s) = \{ t \in \mathbb{L} \mid r < t < s \}$ is open and homeomorphic to $(0, 1)$ in \mathbb{R} , with the homeomorphism being order preserving. Every open set in the long line is a union of such open intervals.
- \mathbb{L} is not second countable, because $\{ \mathcal{O}_\ell \mid \ell \in \mathbb{Y} \}$, with $\mathcal{O}_\ell = \{ (\ell, x) \mid x \in (\frac{1}{4}, \frac{3}{4}) \}$ is an uncountable collection of disjoint nonempty open subsets of \mathbb{L} . The ordinary line \mathbb{R} is homeomorphic to the open interval $(0, 1)$. But the long line is not homeomorphic to any subset of \mathbb{R}^n because it is not second countable.

An atlas for \mathbb{L} is $\mathcal{A} = \{ (\mathcal{U}_{r,s}, \varphi_{r,s}) \mid r, s \in \mathbb{L}, r < s \}$ with $\mathcal{U}_{r,s} = (r, s)$ and $\varphi_{r,s}$ being the homeomorphism referred to above.