Dual Spaces

Definition 1 (Dual Space) Let V be a finite dimensional vector space.

(a) A linear functional on V is a function $\vec{u}^* : V \to \mathbb{R}$ that is linear in the sense that

$$\vec{u}^*(\vec{v}+\vec{w}) = \vec{u}^*(\vec{v}) + \vec{u}^*(\vec{w}) \qquad \text{and} \qquad \vec{u}^*(\alpha\vec{v}) = \alpha\vec{u}^*(\vec{v})$$

for all $\vec{u}, \vec{w} \in V$ and all $\alpha \in \mathbb{R}$.

(b) The dual space V^* of the vector space V is the set of all linear functionals on V. It is itself a vector space with the operations

$$\begin{aligned}(\vec{u}^* + \vec{v}^*)(\vec{w}) &= \vec{u}^*(\vec{w}) + \vec{v}^*(\vec{w}) \\ (\alpha\vec{v}^*)(\vec{w}) &= \alpha\vec{v}^*(\vec{w})\end{aligned}$$

(c) If V has dimension d and has basis $\{\vec{e}_1, \cdots, \vec{e}_d\}$ then, by Problem 1, below, V^* also has dimension d and a basis for V^* (called the dual basis of $\{\vec{e}_1, \cdots, \vec{e}_d\}$) is $\{\vec{e}_1^*, \cdots, \vec{e}_d^*\}$ where, for each $1 \le j \le d$

$$\vec{e}_j^*\big(\alpha_1\vec{e}_1 + \cdots + \alpha_d\vec{e}_d\big) = \alpha_j$$

That is,

$$\vec{e}_j^*(\vec{e}_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases}$$

Problem 1 Let $\{\vec{e}_1, \cdots, \vec{e}_d\}$ be a basis for the vector space V and define, for each $1 \le j \le d$, $\vec{e}_j^*$ as in Definition 1.c. Prove that $\{\vec{e}_1^*, \cdots, \vec{e}_d^*\}$ is a basis for V^*.

Problem 2 Let W be a proper subspace of the finite dimensional vector space V and let $\vec{e}$ be an element of V that is not in W. Prove that there exists a linear functional $\vec{w}^* \in V^*$ such that $\vec{w}^*(\vec{e}) = 1$ and $\vec{w}^*(\vec{w}) = 0$ for all $\vec{w} \in W$.

Problem 3 Let V be a finite dimensional vector space. Each $\vec{v} \in V$ has naturally associated to it the linear functional $\vec{v}^{**} \in (V^*)^*$ (i.e. $\vec{v}^{**}$ is a linear functional on the dual space V^*) defined by

$$\vec{v}^{**}(\vec{w}^*) = \vec{w}^*(\vec{v}) \qquad \text{for all } \vec{w}^* \in V^*$$

Prove that the map $\vec{v} \mapsto \vec{v}^{**}$ is an isomorphism (i.e. 1–1, onto and linear) between the vector spaces V and $(V^*)^*$. Consequently, for finite dimensional vector spaces, one usually thinks of V and $(V^*)^*$ as being the same.

Lemma 2 *Let V be a vector space of dimension $d < \infty$. Let $\vec{v}^* \in V^*$. Its kernel*

$$K = \big\{\ \vec{v} \in V\ \big|\ \vec{v}^*(\vec{v}) = 0\ \big\}$$

is a linear subspace of V. If $\vec{v}^*$ is not the zero functional, then the dimension of K is exactly $d-1$.

Proof: That K is a linear subspace of V is obvious from the linearity of $\vec{v}^*$. Assume that $\vec{v}^*$ is not the zero functional. Then there is some zector $\vec{e}_d \in V$ with $\vec{v}^*(\vec{e}_d) \neq 0$. Choose any $d-1$ additional vectors $\vec{f}_1, \cdots, \vec{f}_{d-1}$ such that $\vec{f}_1, \cdots, \vec{f}_{d-1}, \vec{e}_d$ is a basis for V. For each $1 \le \ell \le d-1$, set

$$\vec{e}_\ell = \vec{f}_\ell - \tfrac{\vec{v}^*(\vec{f}_\ell)}{\vec{v}^*(\vec{e}_d)}\vec{e}_d$$

Then $\vec{e}_1, \cdots, \vec{e}_{d-1}, \vec{e}_d$ is again a basis for V and $\vec{v}^*(\vec{e}_\ell) = 0$ for all $1 \le \ell \le d-1$. Indeed K is exactly the subspace spanned by $\vec{e}_1, \cdots, \vec{e}_{d-1}$ and hence has dimension $d-1$. ■

Definition 3 (transpose) Let V and W be finite dimensional vector spaces with dimensions d_V and d_W, respectively.

(a) A linear map (or linear transformation) from V to W is a function $A : V \to W$ that is linear in the sense that

$$A(\vec{v} + \vec{w}) = A(\vec{v}) + A(\vec{w}) \qquad \text{and} \qquad A(\alpha\vec{v}) = \alpha A(\vec{v})$$

for all $\vec{u}, \vec{w} \in V$ and all $\alpha \in \mathbb{R}$. Often, one writes $A\vec{v}$ in place of $A(\vec{v})$.

(b) If $\{\vec{e}_j\}_{1\le j\le d_V}$ and $\{\vec{f}_j\}_{1\le j\le d_W}$ are bases for V and W respectively, then the matrix $\big[A_{i,j}\big]_{\substack{1\le i\le d_W\\ 1\le j\le d_V}}$ of the linear map $A : V \to W$ is determined by

$$A\vec{e}_j = \sum_{i=1}^{d_W} A_{i,j}\vec{f}_i \qquad \text{for all } 1 \le j \le d_V$$

If $\vec{v} = \sum_{j=1}^{d_V} v_j\vec{e}_j$ then

$$A\vec{v} = \sum_{j=1}^{d_V} v_j A\vec{e}_j = \sum_{j=1}^{d_V}\sum_{i=1}^{d_W} v_j A_{i,j}\vec{f}_i$$

so that the components w_i of the output vector $A\vec{v} = \vec{w} = \sum_{i=1}^{d_W} w_i\vec{f}_i$ are determined by

$$w_i = \sum_{j=1}^{d_V} A_{i,j}v_j \qquad \text{for each } 1 \le i \le d_W$$

(c) The transpose of a linear transformation $A : V \to W$ is the linear transformation $A^t : W^* \to V^*$ determined by

$$(A^t\vec{w}^*)(\vec{v}) = \vec{w}^*(A\vec{v})$$

Note that

- for each fixed $\vec{w}^*$ and A, the right hand side is real valued and linear in $\vec{v}$ so that $A^t\vec{w}^*$ is a legitimate element of V^* and
- the right hand side is linear in $\vec{w}^*$, so that A^t is a linear transformation from W^* to V^*.

- if $\vec{v} = \sum_{i=1}^{d_V} v^i \vec{e}_i$,

$$
\begin{aligned}
(A^t \vec{f}_j^*)(\vec{v}) &= \vec{f}_j^*(A\vec{v}) = \sum_{i=1}^{d_V} v^i \vec{f}_j^*(A\vec{e}_i) = \sum_{i=1}^{d_V}\sum_{\ell=1}^{d_W} v^i A_{\ell,i} \vec{f}_j^*(\vec{f}_\ell) = \sum_{i=1}^{d_V} A_{j,i} v^i \\
&= \sum_{i=1}^{d_V} A_{j,i} \vec{e}_i^*(\vec{v})
\end{aligned}
$$

  we have $A^t \vec{f}_j^* = \sum_{i=1}^{d_V} A_{j,i}\vec{e}_i^*$, so the matrix for the transpose is $A^t_{i,j} = A_{j,i}$, the usual matrix transpose.

Lemma 4 *Let V and W be finite dimensional spaces and let $A : V \to W$ be linear. Denote the transpose of A by $A^t : W^* \to V^*$. Then*

$$
\begin{aligned}
&A^t \text{ is 1–1} \iff A \text{ is onto} \\
&A^t \text{ is onto} \iff A \text{ is 1–1}
\end{aligned}
$$

Proof 1: We first prove that A^t is 1–1 $\iff$ A is onto.

- $A^t\vec{w}^*$ is the zero element of V^* if and only if $(A^t\vec{w}^*)(\vec{v}) = \vec{w}^*(A\vec{v})$ is zero for all $\vec{v} \in V$.
- In the event that A is onto, then $\{ A\vec{v} \mid \vec{v} \in V \} = W$, and $\vec{w}^*(A\vec{v})$ is zero for all $\vec{v} \in V$ if and only if $\vec{w}^*(\vec{w})$ is zero for all $\vec{w} \in W$ which is the case if and only if $\vec{w}^*$ is the zero element of V^*. This proves that A^t is 1–1 when A is onto.
- On the other hand if A is not onto, its range W' is some proper subspace of W. Let $\vec{w}_1, \cdots, \vec{w}_{d_W}$ be a basis for W whose first $d_{W'} < d_W$ vectors form a basis for W'. Then $\vec{w}^* = \vec{w}^*_{d_W}$ (the last linear functional in the dual basis) takes the value 1 when applied to $\vec{w}_{d_W}$ and takes the value 0 when applied to any element of W'. Thus when A is not onto, we can have $(A^t\vec{w}^*_{d_W})(\vec{v}) = \vec{w}^*_{d_W}(A\vec{v}) = \vec{0}$ for all $\vec{v} \in V$ so that $A^t\vec{w}^*_{d_W}$ is the zero element of V^* even though $\vec{w}^*_{d_W}$ is not the zero element of W^*. So A^t is not 1–1.

We now prove that A is 1–1 $\iff$ A^t is onto.

- $A\vec{v}$ is the zero element of W if and only if $\vec{w}^*(A\vec{v}) = (A^t\vec{w}^*)(\vec{v})$ is zero for all $\vec{w}^* \in W^*$.
- In the event that A^t is onto, then $\{ A^t\vec{w}^* \mid \vec{w}^* \in W^* \} = V^*$ and $(A^t\vec{w}^*)(\vec{v})$ is zero for all $\vec{w}^* \in W^*$ if and only if $\vec{v}^*(\vec{v})$ is zero for all $\vec{v}^* \in V^*$, which, in turn, is the case if and only if $\vec{v}$ is the zero element of V. This proves that A is 1–1 when A^t is onto.
- On the other hand if A^t is not onto, its range is some proper subspace of V^*. Let $\vec{v}_1^*, \cdots, \vec{v}_{d'}^*$ be a basis for the range of A^t. We are assuming that $d' < d_{V^*} = d_V$, the dimension of V. Let $\vec{v}_1, \cdots, \vec{v}_{d_V}$ be any basis for V. Then $\vec{v} = \alpha_1\vec{v}_1 + \cdots + \alpha_{d_V}\vec{v}_{d_V}$ has $(A^t\vec{w}^*)(\vec{v}) = 0$ for all $\vec{w}^* \in W^*$ if and only if

$$
0 = \vec{v}_j^*(\alpha_1\vec{v}_1 + \cdots + \alpha_{d_V}\vec{v}_{d_V}) = \sum_{\ell=1}^{d_V} \vec{v}_j^*(\vec{v}_\ell)\,\alpha_\ell \qquad \text{for all } 1 \le j \le d'
$$

  This is a system of $d' < d_V$ homogeneous linear equations in the d_V unknowns $\alpha_1, \cdots, \alpha_{d_V}$. It necessarily has a solution $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{d_V}$ with at least one $\tilde{\alpha}_\ell$ nonzero. Then $\vec{v} = \tilde{\alpha}_1\vec{v}_1 + \cdots + \tilde{\alpha}_{d_V}\vec{v}_{d_V}$ is a nonzero vector with $\vec{w}^*(A\vec{v}) = (A^t\vec{w}^*)(\vec{v}) = 0$ for all $\vec{w}^* \in W^*$ and hence with $A\vec{v} = 0$. So A is not 1–1.

Proof 2: Let $[A_{i,j}]$ be the matrix of A in some bases for V and W. Denote by $\vec{a}_j$, $1 \le j \le d_V$ the column vectors of that matrix. Then A is 1-1 if and only if the column vectors are all linearly independent. That is, if the dimension of the space spanned by the column vectors (i.e. the rank of A) is d_V. The space spanned by the column vectors is exactly the range of A (expressed in the basis of W). So A is onto if and only if the rank of A is d_W. But the rank of a matrix is the same as the rank of its transpose (i.e. the column rank of a matrix is the same as its row rank), so

$$\begin{aligned} A \text{ injective} &\iff \text{rank}(A) = d_V \iff \text{rank}(A^t) = d_{V^*} \iff A^t \text{ surjective} \\ A \text{ surjective} &\iff \text{rank}(A) = d_W \iff \text{rank}(A^t) = d_{W^*} \iff A^t \text{ injective} \end{aligned}$$

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