The Classical Weierstrass Theorem

Theorem (Weierstrass) *If f is a continuous complex valued function on $[a,b]$, then there exists a sequence of polynomials $P_n(x)$ such that*

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a,b]$. If f is real valued, the P_n's may be taken real.

Proof:

Reductions. By scaling and translating the x–axis, we may assume that $[a,b]=[0,1]$. We may also assume, wolog, that $f(0)=f(1)=0$. Once the theorem is proven in this case, apply it to $g(x)=f(x)-f(0)-x[f(1)-f(0)]$. This gives polynomials $\tilde{P}_n$ that converge uniformly to g. Then the polynomials $P_n(x)=\tilde{P}_n(x)+x[f(1)-f(0)]+f(0)$ converge uniformly to f.

Construction of an approximate delta function. Let, for each $n\in\mathbb{N}$, $Q_n(x)=c_n(1-x^2)^n$ where $c_n=\left[\int_{-1}^{1}(1-x^2)^n\ dx\right]^{-1}$. So Q_n is a polynomial that obeys $\int_{-1}^{1}Q_n(x)\ dx=1$ and $0\le Q_n(x)\le c_n$ for all $x\in[-1,1]$. Note that if $n\ge m\ge 1$, $\frac{3}{4}\le c_m\le c_n<\frac{n+1}{2}$ since

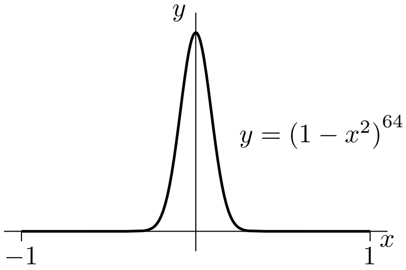
$$\int_{-1}^{1}\left(1-x^2\right)^n\ dx\le\int_{-1}^{1}\left(1-x^2\right)^m\ dx\le\int_{-1}^{1}\left(1-x^2\right)\ dx=\tfrac{4}{3}$$

and

$$\int_{-1}^{1}\left(1-x^2\right)^n\ dx=2\int_{0}^{1}\left(1-x^2\right)^n\ dx>2\int_{0}^{1}\left(1-x\right)^n\ dx=\tfrac{2}{n+1}$$

So $Q_n(0)$ increases with n and is always at least $\frac{3}{4}$. On the other hand, given any δ and any $\varepsilon>0$, there is an N such that for all $n\ge N$ and all $\delta\le|x|\le 1$,

$$0\le Q_n(x)=c_n(1-x^2)^n\le\tfrac{n+1}{2}\left(1-\delta^2\right)^n<\varepsilon$$



Construction of the polynomials. Extend f to the whole real line by defining $f(x)=0$ for all $x\notin[0,1]$. Set

$$P_n(x)=\int_{-1}^{1}f(x+t)Q_n(t)\ dt$$

Since, for $0\le x\le 1$,

$$P_n(x)=\int_{x-1}^{1+x}f(t)Q_n(t-x)\ dt=\int_{0}^{1}f(t)Q_n(t-x)\ dt$$

is a polynomial. Let $\varepsilon'>0$. Since f is uniformly continuous on the whole real line, there is a $\delta>0$ such that $|f(x+t)-f(x)|<\frac{\varepsilon'}{2}$ for all $|t|\le\delta$. Also $M=\sup_{x\in\mathbb{R}}|f(x)|$ is finite. We have also seen that there is an $N\in\mathbb{N}$ such that $0\le Q_n(t)<\frac{\varepsilon'}{8M}$ for all $\delta\le|t|\le 1$ and $n\ge N$. Thus, for all $n\ge N$,

$$\begin{aligned}|P_n(x)-f(x)|&=\Big|\int_{-1}^{1}[f(x+t)-f(x)]Q_n(t)\ dt\Big|\le 2M\int_{-1}^{-\delta}Q_n(t)\ dt+\tfrac{\varepsilon'}{2}\int_{-\delta}^{\delta}Q_n(t)\ dt+2M\int_{\delta}^{1}Q_n(t)\ dt\\&\le 2M\tfrac{\varepsilon'}{8M}+\tfrac{\varepsilon'}{2}+2M\tfrac{\varepsilon'}{8M}=\varepsilon'\end{aligned}$$

■