The Classical Weierstrass Theorem

**Theorem (Weierstrass)** If \( f \) is a continuous complex valued function on \([a, b]\), then there exists a sequence of polynomials \( P_n(x) \) such that

\[
\lim_{n \to \infty} P_n(x) = f(x)
\]

uniformly on \([a, b]\). If \( f \) is real valued, the \( P_n \)'s may be taken real.

**Proof:**

*Reductions.* By scaling and translating the \( x \)-axis, we may assume that \([a, b] = [0, 1]\). We may also assume, \( \text{wolog, that } f(0) = f(1) = 0 \). Once the theorem is proven in this case, apply it to \( g(x) = f(x) - f(0) - x[f(1) - f(0)] \). This gives polynomials \( \tilde{P}_n \) that converge uniformly to \( g \). Then the polynomials \( P_n(x) = \tilde{P}_n(x) + x[f(1) - f(0)] + f(0) \) converge uniformly to \( f \).

*Construction of an approximate delta function.* Let, for each \( n \in \mathbb{N} \), \( Q_n(x) = c_n(1 - x^2)^n \) where \( c_n = \left( \int_{-1}^1 (1 - x^2)^n \, dx \right)^{-1} \). So \( Q_n \) is a polynomial that obeys \( \int_{-1}^1 Q_n(x) \, dx = 1 \) and \( 0 \leq Q_n(x) \leq c_n \) for all \( x \in [-1, 1] \). Note that if \( n \geq m \geq 1 \), \( \frac{3}{4} \leq c_m \leq c_n < \frac{n+1}{2} \) since

\[
\int_{-1}^1 (1 - x^2)^n \, dx \leq \int_{-1}^1 (1 - x^2)^m \, dx \leq \int_{-1}^1 (1 - x^2) \, dx = \frac{2}{3}
\]

and

\[
\int_{-1}^1 (1 - x^2)^n \, dx = 2 \int_{0}^1 (1 - x^2)^n \, dx > 2 \int_{0}^1 (1 - x)^n \, dx = \frac{2}{n+1}
\]

So \( Q_n(0) \) increases with \( n \) and is always at least \( \frac{3}{4} \). On the other hand, given any \( \delta \) and any \( \varepsilon > 0 \), there is an \( N \) such that for all \( n \geq N \) and all \( \delta \leq |x| \leq 1 \),

\[
0 \leq Q_n(x) = c_n(1 - x^2)^n \leq \frac{n+1}{2}(1 - \delta^2)^n < \varepsilon
\]

*Construction of the polynomials.* Extend \( f \) to the whole real line by defining \( f(x) = 0 \) for all \( x \notin [0, 1] \). Set

\[
P_n(x) = \int_{-1}^1 f(x + t)Q_n(t) \, dt
\]

Since, for \( 0 \leq x \leq 1 \),

\[
P_n(x) = \int_{x-1}^{1+x} f(t)Q_n(t-x) \, dt = \int_{0}^1 f(t)Q_n(t-x) \, dt
\]

is a polynomial. Let \( \varepsilon' > 0 \). Since \( f \) is uniformly continuous on the whole real line, there is a \( \delta > 0 \) such that \(|f(x + t) - f(x)| < \frac{\varepsilon'}{2} \) for all \(|t| \leq \delta \). Also \( M = \sup_{x \in \mathbb{R}} |f(x)| \) is finite. We have also seen that there is an \( N \in \mathbb{N} \) such that \( 0 \leq Q_n(t) < \frac{\varepsilon'}{8M} \) for all \( \delta \leq |t| \leq 1 \) and \( n \geq N \). Thus, for all \( n \geq N \),

\[
|P_n(x) - f(x)| = \left| \int_{-1}^1 [f(x + t) - f(x)]Q_n(t) \, dt \right| \leq 2M \int_{-1}^{-\delta} Q_n(t) \, dt + \frac{\varepsilon'}{2} \int_{-\delta}^\delta Q_n(t) \, dt + 2M \int_{\delta}^1 Q_n(t) \, dt \leq 2M \frac{\varepsilon'}{8M} + \frac{\varepsilon'}{2} + 2M \frac{\varepsilon'}{8M} = \varepsilon'
\]

\[\square\]

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