

# A Lie Group

These notes introduce  $SU(2)$  as an example of a compact Lie group.

## The Definition

The definition of  $SU(2)$  is

$$SU(2) = \{ A \mid A \text{ a } 2 \times 2 \text{ complex matrix, } \det A = 1, AA^* = A^*A = \mathbb{1} \}$$

In the name  $SU(2)$ , the “S” stands for “special” and refers to the condition  $\det A = 1$  and the “U” stands for “unitary” and refers to the conditions  $AA^* = A^*A = \mathbb{1}$ . The adjoint matrix  $A^*$  is the complex conjugate of the transpose matrix. That is,

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^* = \begin{bmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{bmatrix}$$

Define the inner product on  $\mathbb{C}^2$  by

$$\left\langle \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2$$

The adjoint matrix was defined so that

$$\sum_{i,j=1}^2 A_{i,j} a_j \bar{b}_i = \left\langle A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, A^* \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle = \sum_{i,j=1}^2 a_j \overline{A_{j,i}^* b_i}$$

Thus the condition  $A^*A = \mathbb{1}$  is equivalent to

$$\begin{aligned} \left\langle A^*A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle \quad \text{for all } \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{C}^2 \\ \iff \left\langle A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, A \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle \quad \text{for all } \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{C}^2 \end{aligned}$$

Hence  $SU(2)$  is the set of  $2 \times 2$  complex matrices that have determinant one and preserve the inner product on  $\mathbb{C}^2$ . (Recall that, for square matrices,  $A^*A = \mathbb{1}$  is equivalent to  $A^{-1} = A^*$ , which in turn is equivalent to  $AA^* = \mathbb{1}$ .) By the polarization identity (Problem Set V, #3), preservation of the inner product is equivalent to preservation of the norm

$$\left\| A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right\| \quad \text{for all } \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2$$

Clearly  $\mathbb{1} \in SU(2)$ . If  $A, B \in SU(2)$ , then  $\det(AB) = \det(A)\det(B) = 1$  and  $(AB)(AB)^* = ABB^*A^* = A\mathbb{1}A^* = \mathbb{1}$  so that  $AB \in SU(2)$ . Also, if  $A \in SU(2)$ , then  $A^{-1} = A^* \in SU(2)$ . So  $SU(2)$  is a group. We may also view  $SU(2)$  as a subset of  $\mathbb{C}^4$ . Then  $SU(2)$  inherits a topology from  $\mathbb{C}^4$ , so that  $SU(2)$  is a topological group.

## The Pauli Matrices

The matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are called the Pauli matrices. They obey  $\sigma_\ell = \sigma_\ell^*$  for all  $\ell = 1, 2, 3$  and also obey

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1} \quad \sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3 \quad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1 \quad \sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2 \quad (1)$$

Set, for each  $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ , the matrix

$$\vec{a} \cdot \vec{\sigma} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$$

Then the product rules (1) can be written

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b} \mathbb{1} + i\vec{a} \times \vec{b} \cdot \vec{\sigma} \quad (2)$$

I claim that any  $2 \times 2$  complex matrix has a unique representation of the form  $a_0\mathbb{1} + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3$  for some  $a_0, a_1, a_2, a_3 \in \mathbb{C}$ . This is easy to see. Since

$$a_0\mathbb{1} + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{bmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{bmatrix}$$

we have that

$$a_0\mathbb{1} + ia_1\sigma_1 + ia_2\sigma_2 + ia_3\sigma_3 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \iff a_0 = \frac{\alpha + \delta}{2}, a_1 = \frac{\beta + \gamma}{2i}, a_2 = \frac{\beta - \gamma}{2}, a_3 = \frac{\alpha - \delta}{2i}$$

**Lemma.**

$$SU(2) = \{ x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma} \mid (x_0, \vec{x}) \in \mathbb{R}^4, x_0^2 + \|\vec{x}\|^2 = 1 \}$$

**Proof:** Let  $A$  be any  $2 \times 2$  complex matrix and write  $A = a_0\mathbb{1} + i\vec{a} \cdot \vec{\sigma}$  with  $\vec{a} = (a_1, a_2, a_3)$ . Then by (2)

$$\begin{aligned} AA^* &= (a_0\mathbb{1} + i\vec{a} \cdot \vec{\sigma})(\overline{a_0}\mathbb{1} - i\vec{\bar{a}} \cdot \vec{\sigma}) \\ &= |a_0|^2\mathbb{1} + i\overline{a_0}\vec{a} \cdot \vec{\sigma} - ia_0\vec{\bar{a}} \cdot \vec{\sigma} + \vec{a} \cdot \vec{\bar{a}}\mathbb{1} + i\vec{a} \times \vec{\bar{a}} \cdot \vec{\sigma} \\ &= (|a_0|^2 + \|\vec{a}\|^2)\mathbb{1} + i(\overline{a_0}\vec{a} - a_0\vec{\bar{a}} + \vec{a} \times \vec{\bar{a}}) \cdot \vec{\sigma} \end{aligned}$$

Hence

$$AA^* = \mathbb{1} \iff |a_0|^2 + \|\vec{a}\|^2 = 1, \overline{a_0}\vec{a} - a_0\vec{\bar{a}} + \vec{a} \times \vec{\bar{a}} = 0$$

First, suppose that  $\vec{a} \neq \vec{0}$ . Since  $\vec{a} \times \vec{\bar{a}}$  is orthogonal to both  $\vec{a}$  and  $\vec{\bar{a}}$ , the equation  $\overline{a_0}\vec{a} - a_0\vec{\bar{a}} + \vec{a} \times \vec{\bar{a}} = 0$  can be satisfied only if  $\vec{a} \times \vec{\bar{a}} = 0$ . That is, only if  $\vec{a}$  and  $\vec{\bar{a}}$  are parallel. Since  $\vec{a}$  and  $\vec{\bar{a}}$  have the same length, this is the case only if  $\vec{\bar{a}} = e^{-2i\theta}\vec{a}$  for some real number  $\theta$ . This is equivalent to  $\overline{e^{-i\theta}\vec{a}} = e^{-i\theta}\vec{a}$  which says that  $\vec{x} = e^{-i\theta}\vec{a}$  is real. Subbing  $\vec{a} = e^{i\theta}\vec{x}$  back into  $\overline{a_0}\vec{a} - a_0\vec{\bar{a}} + \vec{a} \times \vec{\bar{a}} = 0$  gives

$$e^{i\theta}\overline{a_0}\vec{x} - e^{-i\theta}a_0\vec{x} = 0$$

This forces  $a_0 = e^{i\theta}x_0$  for some real  $x_0$ . If  $\vec{a} = \vec{0}$ , we may still choose  $\theta$  so that  $a_0 = e^{i\theta}x_0$ . We have now shown that

$$AA^* = \mathbb{1} \iff A = e^{i\theta}(x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma}) \text{ for some } (x_0, \vec{x}) \in \mathbb{R}^4 \text{ with } |x_0|^2 + \|\vec{x}\|^2 = 1 \text{ and some } \theta \in \mathbb{R}$$

Since

$$\det A = \det e^{i\theta}(x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma}) = \det e^{i\theta} \begin{bmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{bmatrix} = e^{2i\theta}(x_0^2 + x_1^2 + x_2^2 + x_3^2) = e^{2i\theta}$$

we have that  $\det A = 1$  if and only if  $e^{i\theta} = \pm 1$ . If  $e^{i\theta} = -1$ , we can absorb the  $-1$  into  $(x_0, \vec{x})$ . ■

As consequences of this Lemma we have that  $SU(2)$  is

- homeomorphic to  $S^3$ , the unit sphere in  $\mathbb{R}^4$
- connected
- simply connected (meaning that every continuous closed curve in  $SU(2)$  can be continuously deformed to a point)
- is a  $C^\infty$  manifold (meaning, roughly speaking, that in a neighbourhood of each point, we may choose three of  $x_0, x_1, x_2, x_3$  as coordinates, with the fourth then determined as a  $C^\infty$  function of the chosen three)

A topological group that is also a  $C^\infty$  manifold (with the maps  $(a, b) \mapsto ab$  and  $a \mapsto a^{-1}$   $C^\infty$  when expressed in local coordinates) is called a Lie Group.

### The Connection between $SU(2)$ and $SO(3)$

Define

$$M : \mathbb{R}^3 \rightarrow V = \{ \vec{a} \cdot \vec{\sigma} \mid \vec{a} \in \mathbb{R}^3 \} \subset \{2 \times 2 \text{ complex matrices}\}$$

$$\vec{a} \mapsto M(\vec{a}) = \vec{a} \cdot \vec{\sigma}$$

This is a linear bijection between  $\mathbb{R}^3$  and  $V$ . (In fact  $V$  is the space of all  $2 \times 2$  traceless, self-adjoint matrices.) Each  $U \in SU(2)$  determines a linear map  $S(U)$  on  $\mathbb{R}^3$  by

$$M(S(U)\vec{a}) = UM(\vec{a})U^{-1} \quad (3)$$

The right hand side is clearly linear in  $\vec{a}$ . But it is not so clear that  $UM(\vec{a})U^{-1}$  is in  $V$ , that is, of the form  $M(\vec{b})$ . To check this, we let  $U = x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma}$  with  $(x_0, \vec{x}) \in \mathbb{R}^4$  obeying  $\|x_0\|^2 + \|\vec{x}\|^2 = 1$  and compute  $UM(\vec{a})U^{-1} = UM(\vec{a})U^*$  explicitly. Applying (2) twice

$$\begin{aligned} UM(\vec{a})U^{-1} &= (x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma})(\vec{a} \cdot \vec{\sigma})(x_0\mathbb{1} - i\vec{x} \cdot \vec{\sigma}) \\ &= (x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma})(x_0\vec{a} \cdot \vec{\sigma} - i\vec{a} \cdot \vec{x}\mathbb{1} + \vec{a} \times \vec{x} \cdot \vec{\sigma}) \\ &= x_0^2\vec{a} \cdot \vec{\sigma} - ix_0\vec{a} \cdot \vec{x}\mathbb{1} + x_0\vec{a} \times \vec{x} \cdot \vec{\sigma} \\ &\quad + ix_0\vec{x} \cdot \vec{a}\mathbb{1} - x_0\vec{x} \times \vec{a} \cdot \vec{\sigma} + \vec{a} \cdot \vec{x}\vec{x} \cdot \vec{\sigma} + i\vec{x} \cdot (\vec{a} \times \vec{x})\mathbb{1} - \vec{x} \times (\vec{a} \times \vec{x}) \cdot \vec{\sigma} \\ &= x_0^2\vec{a} \cdot \vec{\sigma} + 2x_0\vec{a} \times \vec{x} \cdot \vec{\sigma} + \vec{a} \cdot \vec{x}\vec{x} \cdot \vec{\sigma} - \vec{x} \times (\vec{a} \times \vec{x}) \cdot \vec{\sigma} \end{aligned}$$

since  $\vec{x}$  is perpendicular to  $(\vec{a} \times \vec{x})$ . Using  $\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}$ ,

$$\begin{aligned} UM(\vec{a})U^{-1} &= x_0^2\vec{a} \cdot \vec{\sigma} + 2x_0\vec{a} \times \vec{x} \cdot \vec{\sigma} + \vec{a} \cdot \vec{x}\vec{x} \cdot \vec{\sigma} - \|\vec{x}\|^2\vec{a} \cdot \vec{\sigma} + \vec{a} \cdot \vec{x}\vec{x} \cdot \vec{\sigma} \\ &= (x_0^2 - \|\vec{x}\|^2)\vec{a} \cdot \vec{\sigma} + 2x_0\vec{a} \times \vec{x} \cdot \vec{\sigma} + 2\vec{a} \cdot \vec{x}\vec{x} \cdot \vec{\sigma} \end{aligned}$$

This shows, not only that  $UM(\vec{a})U^{-1} \in V$ , but also that, for  $U = x_0\mathbb{1} + i\vec{x} \cdot \vec{\sigma}$ ,

$$S(U)\vec{a} = (x_0^2 - \|\vec{x}\|^2)\vec{a} - 2x_0\vec{x} \times \vec{a} + 2\vec{a} \cdot \vec{x}\vec{x}$$

In fact, we can exactly identify the geometric operation that  $S(U)$  implements. If  $U = \pm\mathbb{1}$ , that is  $\vec{x} = \vec{0}$ , then it is obvious from (3) that  $S(U)\vec{a} = \vec{a}$  for all  $\vec{a} \in \mathbb{R}^3$ . That is, both  $S(\mathbb{1})$  and  $S(-\mathbb{1})$  are the identity map on  $\mathbb{R}^3$ . If  $\vec{x} \neq \vec{0}$ , there is a unique angle  $0 < \theta < 2\pi$  and a unique unit vector  $\hat{e}$  such that  $x_0 = \cos(\frac{\theta}{2})$  and  $\vec{x} = -\sin(\frac{\theta}{2})\hat{e}$ . If  $\vec{a}$  happens to be parallel to  $\vec{x}$ , that is,  $\vec{a} = c\vec{x}$ ,

$$S(U)\vec{a} = (x_0^2 - \|\vec{x}\|^2)\vec{a} + 2\vec{a} \cdot \vec{x}\vec{x} = (x_0^2 + \|\vec{x}\|^2)\vec{a} = \vec{a}$$

So  $S(U)$  leaves the axis  $\vec{x}$  invariant. If  $\vec{a}$  is not parallel to  $\vec{x}$ , set

$$\hat{k} = \hat{e} \quad \hat{i} = \frac{\vec{a} - \vec{a} \cdot \hat{k} \hat{k}}{\|\vec{a} - \vec{a} \cdot \hat{k} \hat{k}\|} \quad \hat{j} = \hat{k} \times \hat{i}$$

This is an orthonormal basis for  $\mathbb{R}^3$ . Since  $\vec{a}$  is a linear combination of  $\hat{i}$  and  $\hat{k}$ ,

$$\vec{a} = \vec{a} \cdot \hat{k} \hat{k} + \vec{a} \cdot \hat{i} \hat{i}$$

In terms of this notation

$$\begin{aligned} S(U)\vec{a} &= \cos(\theta) \vec{a} + \sin(\theta) \hat{e} \times \vec{a} + 2 \sin^2\left(\frac{\theta}{2}\right) \vec{a} \cdot \hat{e} \hat{e} \\ &= \vec{a} \cdot \hat{e} \hat{e} + \cos(\theta) (\vec{a} - \vec{a} \cdot \hat{e} \hat{e}) + \sin(\theta) \hat{e} \times \vec{a} \end{aligned}$$

Since

$$\begin{aligned} \vec{a} - \vec{a} \cdot \hat{e} \hat{e} &= \vec{a} - \vec{a} \cdot \hat{k} \hat{k} = \vec{a} \cdot \hat{i} \hat{i} \\ \hat{e} \times \vec{a} &= \hat{k} \times \vec{a} = \hat{k} \times (\vec{a} \cdot \hat{k} \hat{k} + \vec{a} \cdot \hat{i} \hat{i}) = \vec{a} \cdot \hat{i} \hat{j} \end{aligned}$$

we have

$$S(U)\vec{a} = \vec{a} \cdot \hat{k} \hat{k} + \cos(\theta) \vec{a} \cdot \hat{i} \hat{i} + \sin(\theta) \vec{a} \cdot \hat{i} \hat{j}$$

In particular

$$S(U)\hat{k} = \hat{k} \quad S(U)\hat{i} = \cos(\theta) \hat{i} + \sin(\theta) \hat{j}$$

This is exactly the rotation of  $\vec{a}$  about the axis  $\hat{k} = \hat{e}$  (the  $\hat{k}$  component of  $\vec{a}$  is unchanged) by an angle  $\theta$  (the part of  $\vec{a}$  perpendicular to  $\hat{k}$  has changed by a rotation by  $\theta$  as in  $\mathbb{R}^2$ ). This shows that

$$S : SU(2) \rightarrow SO(3)$$

that  $S$  is surjective and that  $S(U) = \mathbb{1}_3$ , the identity map on  $\mathbb{R}^3$ , if and only if  $U = \pm \mathbb{1}$ . Also, by (3),

$$M(S(UU')\vec{a}) = UU'M(\vec{a})U'^{-1}U^{-1} = U M(S(U')\vec{a}) U^{-1} = M(S(U)S(U')\vec{a})$$

so that  $S(UU') = S(U)S(U')$  and  $S$  is a homomorphism. It is not injective, since  $S(-\mathbb{1}) = S(\mathbb{1})$ . Indeed  $S$  is a two to one map since

$$S(U) = S(\tilde{U}) \iff S(U)S(\tilde{U})^{-1} = \mathbb{1}_3 \iff S(U\tilde{U}^{-1}) = \mathbb{1}_3 \iff U\tilde{U}^{-1} = \pm \mathbb{1} \iff U = \pm \tilde{U}$$

We have now shown that  $SO(3)$  is isomorphic to  $SU(2)/\{\mathbb{1}, -\mathbb{1}\}$ .

## The Haar Measure

Recall that

$$SU(2) = \{ x_0 \mathbb{1} + i\vec{x} \cdot \vec{\sigma} \mid (x_0, \vec{x}) \in \mathbb{R}^4, x_0^2 + \|\vec{x}\|^2 = 1 \}$$

For all  $x_1^2 + x_2^2 + x_3^2 < 1$ ,  $x_0 > 0$ , we can use  $\vec{x}$  as coordinates with  $x_0(\vec{x}) = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}$ . For all  $x_1^2 + x_2^2 + x_3^2 < 1$ ,  $x_0 < 0$ , we can use  $\vec{x}$  as coordinates with  $x_0(\vec{x}) = -\sqrt{1 - x_1^2 - x_2^2 - x_3^2}$ . This leaves only  $x_1^2 + x_2^2 + x_3^2 = 1$ ,  $x_0 = 0$ . We could cover this using other components as coordinates, but as this is a set of measure zero, we won't bother. Denote

$$\begin{aligned} \gamma_+(\vec{x}) &= \sqrt{1 - x_1^2 - x_2^2 - x_3^2} \mathbb{1} + i\vec{x} \cdot \vec{\sigma} \\ \gamma_-(\vec{x}) &= -\sqrt{1 - x_1^2 - x_2^2 - x_3^2} \mathbb{1} + i\vec{x} \cdot \vec{\sigma} \end{aligned}$$

We shall now find two functions  $\Delta_+(\vec{x})$  and  $\Delta_-(\vec{x})$  such that, for all continuous functions  $f$  on  $SU(2)$

$$\int_{SU(2)} f(\gamma) d\mu(\gamma) = \iiint_{\|\vec{x}\| < 1} f(\gamma_+(\vec{x})) \Delta_+(\vec{x}) d^3\vec{x} + \iiint_{\|\vec{x}\| < 1} f(\gamma_-(\vec{x})) \Delta_-(\vec{x}) d^3\vec{x}$$

where  $\mu$  is the Haar measure on  $SU(2)$ .

Define  $\vec{z}_+(\vec{y}, \vec{x})$  and  $\vec{z}_-(\vec{y}, \vec{x})$  by

$$\gamma_+(\vec{z}_+(\vec{y}, \vec{x})) = \gamma_+(\vec{y})\gamma_+(\vec{x}) \quad \gamma_-(\vec{z}_-(\vec{y}, \vec{x})) = \gamma_-(\vec{y})\gamma_+(\vec{x})$$

If you multiply an element of the interior of the upper hemisphere of  $SU(2)$  (like  $\gamma_+(\vec{y})$  with  $\|\vec{y}\| < 1$ ) by an element of  $SU(2)$  that is sufficiently close to the identity (like  $\gamma_+(\vec{x})$  with  $\|\vec{x}\| \ll 1$ ) you end up with another element of the interior of the upper hemisphere. Similarly, if you multiply an element of the interior of the lower hemisphere of  $SU(2)$  (like  $\gamma_-(\vec{y})$  with  $\|\vec{y}\| < 1$ ) by an element of  $SU(2)$  that is sufficiently close to the identity (like  $\gamma_+(\vec{x})$  with  $\|\vec{x}\| \ll 1$ ) you end up with another element of the interior of the lower hemisphere. Thus both  $\vec{z}_+(\vec{y}, \vec{x})$  and  $\vec{z}_-(\vec{y}, \vec{x})$  make sense for all  $\vec{y}$  with  $\|\vec{y}\| < 1$  provided  $\|\vec{x}\|$  is sufficiently small (depending on  $\vec{y}$ ). By the argument of Example 5.ii of the notes ‘‘Haar Measure’’

$$\Delta_+(\vec{0}) = \Delta_+(\vec{y}) \left| \det \left[ \frac{\partial z_{+i}}{\partial x_j}(\vec{y}, \vec{0}) \right]_{1 \leq i, j \leq 3} \right| \quad \Delta_-(\vec{0}) = \Delta_-(\vec{y}) \left| \det \left[ \frac{\partial z_{-i}}{\partial x_j}(\vec{y}, \vec{0}) \right]_{1 \leq i, j \leq 3} \right|$$

This will determine both  $\Delta_+(\vec{y})$  and  $\Delta_-(\vec{y})$  up to the constant  $\Delta_+(\vec{0})$ . The latter will be determined by the requirement that the measure have total mass one.

We first find  $\vec{z}_+$  and  $\vec{z}_-$ . By (2),

$$(y_0 \mathbb{1} + i\vec{y} \cdot \vec{\sigma})(x_0 \mathbb{1} + i\vec{x} \cdot \vec{\sigma}) = y_0 x_0 \mathbb{1} + iy_0 \vec{x} \cdot \vec{\sigma} + ix_0 \vec{y} \cdot \vec{\sigma} - \vec{x} \cdot \vec{y} \mathbb{1} - i(\vec{y} \times \vec{x}) \cdot \vec{\sigma}$$

Thus

$$\begin{aligned} \vec{z}_+(\vec{y}, \vec{x}) &= y_0 \vec{x} + x_0 \vec{y} - \vec{y} \times \vec{x} \quad \text{with} \quad y_0 = \sqrt{1 - \|\vec{y}\|^2} \quad \text{and} \quad x_0 = \sqrt{1 - \|\vec{x}\|^2} \\ \vec{z}_-(\vec{y}, \vec{x}) &= y_0 \vec{x} + x_0 \vec{y} - \vec{y} \times \vec{x} \quad \text{with} \quad y_0 = -\sqrt{1 - \|\vec{y}\|^2} \quad \text{and} \quad x_0 = \sqrt{1 - \|\vec{x}\|^2} \end{aligned}$$

Next we compute the matrices of partial derivatives. Observe that

$$\begin{aligned} \frac{\partial}{\partial x_j} y_0 x_i &= y_0 \delta_{i,j} \\ \frac{\partial}{\partial x_j} \sqrt{1 - \|\vec{x}\|^2} \vec{y} \Big|_{\vec{x}=0} &= \frac{-x_j}{\sqrt{1 - \|\vec{x}\|^2}} \vec{y} \Big|_{\vec{x}=0} = \vec{0} \\ -\frac{\partial}{\partial x_j} \vec{y} \times \vec{x} &= \frac{\partial}{\partial x_j} (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) = \begin{cases} (0, -y_3, y_2) & \text{if } j = 1 \\ (y_3, 0, -y_1) & \text{if } j = 2 \\ (-y_2, y_1, 0) & \text{if } j = 3 \end{cases} \end{aligned}$$

Hence, with  $y_0 = \pm \sqrt{1 - \|\vec{y}\|^2}$  for  $\vec{z}_\pm$ ,

$$\begin{aligned} \det \left[ \frac{\partial z_{\pm i}}{\partial x_j}(\vec{y}, \vec{0}) \right]_{1 \leq i, j \leq 3} &= \det \begin{bmatrix} y_0 & y_3 & -y_2 \\ -y_3 & y_0 & y_1 \\ y_2 & -y_1 & y_0 \end{bmatrix} \\ &= y_0^3 + y_3 y_1 y_2 - y_2 y_3 y_1 - (-y_0 y_1^2 - y_0 y_3^2 - y_0 y_2^2) \\ &= y_0 (y_0^2 + y_1^2 + y_2^2 + y_3^2) \\ &= y_0 \end{aligned}$$

Thus

$$\Delta_+(\vec{y}) = \Delta_-(\vec{y}) = \frac{\Delta_+(\vec{0})}{\sqrt{1 - y_1^2 - y_2^2 - y_3^2}}$$

The constant  $\Delta_+(\vec{0})$  is determined by the requirement that

$$1 = \iiint_{\|\vec{y}\| < 1} \Delta_+(\vec{y}) d^3 \vec{y} + \iiint_{\|\vec{y}\| < 1} \Delta_-(\vec{y}) d^3 \vec{y} = 2\Delta_+(\vec{0}) \iiint_{\|\vec{y}\| < 1} \frac{1}{\sqrt{1 - y_1^2 - y_2^2 - y_3^2}} d^3 \vec{y}$$

Switching to conventional spherical coordinates

$$1 = 2\Delta_+(\vec{0}) \int_0^1 d\rho \int_0^{2\pi} d\theta \int_0^\pi d\varphi \rho^2 \sin \varphi \frac{1}{\sqrt{1 - \rho^2}} = 8\pi \Delta_+(\vec{0}) \int_0^1 \frac{\rho^2}{\sqrt{1 - \rho^2}} d\rho$$

Now making the change of variables  $\rho = \sin \alpha$

$$1 = 8\pi \Delta_+(\vec{0}) \int_0^{\pi/2} \frac{\sin^2 \alpha}{\cos \alpha} \cos \alpha d\alpha = 8\pi \Delta_+(\vec{0}) \int_0^{\pi/2} \sin^2 \alpha d\alpha = 2\pi^2 \Delta_+(\vec{0})$$

$$\text{and } \Delta_+(\vec{x}) = \Delta_-(\vec{x}) = \frac{1}{2\pi^2 \sqrt{1 - x_1^2 - x_2^2 - x_3^2}}.$$

This is in fact, aside from a constant factor used to normalize the mass of the measure to one, the standard measure on the sphere  $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$  that is inherited from the standard Lebesgue measure on  $\mathbb{R}^4$ . Recall that the standard surface measure on the surface  $z = f(x, y)$  is  $\sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2} dx dy$ . This is derived in second year Calculus courses by cutting up the surface into tiny parallelograms and computing the area of each parallelogram. This same derivation applied to  $z = f(x_1, x_2, x_3)$  gives  $\sqrt{1 + f_{x_1}(\vec{x})^2 + f_{x_2}(\vec{x})^2 + f_{x_3}(\vec{x})^2} d^3 \vec{x}$ . If  $f(\vec{x}) = \pm \sqrt{1 - x_1^2 - x_2^2 - x_3^2}$  then

$$1 + f_{x_1}(\vec{x})^2 + f_{x_2}(\vec{x})^2 + f_{x_3}(\vec{x})^2 = 1 + \frac{x_1^2 + x_2^2 + x_3^2}{1 - x_1^2 - x_2^2 - x_3^2} = \frac{1}{1 - x_1^2 - x_2^2 - x_3^2}$$

so

$$\sqrt{1 + f_{x_1}(\vec{x})^2 + f_{x_2}(\vec{x})^2 + f_{x_3}(\vec{x})^2} d^3 \vec{x} = \frac{1}{\sqrt{1 - x_1^2 - x_2^2 - x_3^2}} d^3 \vec{x}$$

as desired.