

Theorem (Riesz Representation Theorem) *Let X be a locally compact Hausdorff space and $\Lambda : C_0(X) \rightarrow \mathbb{R}$ or \mathbb{C} be a positive linear functional. Then there exists a σ -algebra^{step 7} \mathcal{M} in X , which contains all Borel sets^{step 7}, and a unique measure μ on \mathcal{M} such that*

- a) $\Lambda f = \int_X f d\mu \quad \forall f \in C_0(X) \quad \text{step 10}$
- b) $\mu(K) < \infty \quad \forall \text{ compact } K \subset X \quad \text{step 2}$
- c) *If $E \in \mathcal{M}$ then*

$$\mu(E) = \inf \{ \mu(V) \mid E \subset V, V \text{ open} \} \quad \text{definition}$$

- d) *If E is open^{step 3} or if $E \in \mathcal{M}$ with $\mu(E) < \infty$ ^{step 8} then*

$$\mu(E) = \sup \{ \mu(K) \mid K \subset E, K \text{ compact} \}$$

- e) *$\langle X, \mathcal{M}, \mu \rangle$ is a complete measure space. That is, if $E \in \mathcal{M}$, $A \subset E$, $\mu(E) = 0 \Rightarrow A \in \mathcal{M}$ step 8 and definition of \mathcal{M}_F*

Formulae for μ^*

$$V \text{ open} \quad \Rightarrow \mu^*(V) = \sup \{ \Lambda f \mid f \prec V \} \quad \text{definition}$$

$$E \text{ arbitrary} \quad \Rightarrow \mu^*(E) = \inf \{ \mu^*(V) \mid V \supset E, V \text{ open} \} \quad \text{definition}$$

$$K \text{ compact} \quad \Rightarrow \mu^*(K) = \inf \{ \Lambda f \mid f \succ K \} \quad \text{after step 2}$$

$$V \text{ open} \quad \Rightarrow \mu^*(V) = \sup \{ \mu^*(K) \mid K \subset V, K \text{ compact} \} \quad \text{after step 3}$$

$$E \in \mathcal{M}_F \quad \Rightarrow \mu^*(E) = \sup \{ \mu^*(K) \mid K \subset E, K \text{ compact} \} \quad \text{definition}$$

Outline

Definitions:

- a) For all open $V \subset X$, define $\mu^*(V) = \sup \{ \int f \mid f \prec V \}$.
- b) For all $E \subset X$, define $\mu^*(E) = \inf \{ \mu^*(V) \mid E \subset V, V \text{ open} \}$.
- c) $\mathcal{M}_F = \{ E \subset X \mid \mu^*(E) < \infty \text{ and } \mu^*(E) = \sup \{ \mu^*(K) \mid K \subset E, K \text{ compact} \} \}$
- d) $\mathcal{M} = \{ E \subset X \mid E \cap K \in \mathcal{M}_F \text{ for all compact } K \}$
- e) μ is the restriction of μ^* to \mathcal{M} .

Step 1. μ^* is an outer measure.

Step 2. K compact $\implies K \in \mathcal{M}_F$ and $\mu^*(K) = \inf \{ \int f \mid K \prec f \}$

Step 3. V open $\implies \mu^*(V) = \sup \{ \mu^*(K) \mid K \subset V, K \text{ compact} \}$

Step 4. If $E_i \in \mathcal{M}_F$, $1 \leq i < \infty$ pairwise disjoint

$$\implies \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i).$$

If, in addition, $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) < \infty$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}_F$.

Step 5. If $E \in \mathcal{M}_F$ and $\varepsilon > 0$, then there exists K , compact and V , open, such that $K \subset E \subset V$ and $\mu^*(V \setminus K) < \varepsilon$.

Step 6. If $A, B \in \mathcal{M}_F$, then $A \setminus B, A \cup B, A \cap B \in \mathcal{M}_F$

Step 7. \mathcal{M} is a σ -algebra and contains all Borel sets.

Step 8. $\mathcal{M}_F = \{ E \in \mathcal{M} \mid \mu^*(E) < \infty \}$

Step 9. μ is a measure

Step 10. $f \in C_0(X) \implies \int f = \int_X f \, d\mu$.