

Math 421/510 Problem Set X

Due March 27

1) Examine the proof of the Riesz representation theorem and prove the following two statements:

- a) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$, even if E_1 and E_2 are not in \mathcal{M} .
- b) If $E \in \mathcal{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \dots$ where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

2) Let X be a compact Hausdorff space. Let F be a bounded linear functional on $C(X)$. Suppose that G and H are two positive linear functionals on $C(X)$ with $F = G - H$ and $G(1) + H(1) \leq \|F\|$. Prove that $G = F_+$ and $H = F_-$. Hint: Use the definition of F_+ to show that $G - F_+$ is a positive linear functional.

3) Let Γ be a compact group with Haar measure μ . Let $D^{(\alpha)}$ and $D^{(\beta)}$ be irreducible representations of Γ by $d_\alpha \times d_\alpha$ and $d_\beta \times d_\beta$ matrices, respectively. Prove that, for all $1 \leq i, j \leq d_\alpha$ and $1 \leq k, \ell \leq d_\beta$

$$\int D_{i,j}^{(\alpha)}(\gamma) \overline{D_{k,\ell}^{(\beta)}(\gamma)} d\mu(\gamma) = \frac{1}{d_\alpha} \delta_{i,k} \delta_{j,\ell} \begin{cases} 0 & \text{if } D^{(\alpha)} \text{ and } D^{(\beta)} \text{ are not equivalent} \\ 1 & \text{if } D^{(\alpha)} = D^{(\beta)} \end{cases}$$

4) Let $\chi_\alpha(\gamma) = \text{Tr } D^{(\alpha)}(\gamma)$ and $\chi_\beta(\gamma) = \text{Tr } D^{(\beta)}(\gamma)$ where $D^{(\alpha)}$ and $D^{(\beta)}$ are the representations of Problem 3. Prove that

$$\int \chi_\alpha(\gamma) \overline{\chi_\beta(\gamma)} d\mu(\gamma) \begin{cases} 0 & \text{if } D^{(\alpha)} \text{ and } D^{(\beta)} \text{ are not equivalent} \\ 1 & \text{if } D^{(\alpha)} = D^{(\beta)} \end{cases}$$