**Math 421/510 Problem Set I**

Due January 16

1a) Let $X$ be a set. Is there a metric on $X$ so that the associated topology is trivial? discrete?

b) Let $X$ be a space with the trivial topology. Find all continuous functions $f : X \to \mathbb{R}$. Find all continuous functions $g : \mathbb{R} \to X$.

c) Let $X$ be a space with the discrete topology. Find all continuous functions $f : X \to \mathbb{R}$. Find all continuous functions $g : \mathbb{R} \to X$.

2) Let $X, Y$ be topological spaces.

a) Prove that if a subset $F \subset X$ is closed then

$$\left\{ x_n \;\middle|\; n \in \mathbb{N} \right\} \subset F,\ x = \lim_{n\to\infty} x_n \Rightarrow x \in F$$

b) Prove that if $f : X \to Y$ is continuous then

$$x = \lim_{n\to\infty} x_n \Rightarrow f(x) = \lim_{n\to\infty} f(x_n)$$

c) A base at $x \in X$ is a family $\mathcal{B}_x$ of open sets with the property that if $\mathcal{O}$ is an open set containing $x$ then there is a $B \in \mathcal{B}_x$ such that $x \in B \subset \mathcal{O}$. Suppose that $X$ has a countable base at each $x \in X$. (For example, every metric space has a countable base at each of its points.) Prove that $F \subset X$ is closed if and only if

$$\left\{ x_n \;\middle|\; n \in \mathbb{N} \right\} \subset F,\ x = \lim_{n\to\infty} x_n \Rightarrow x \in F$$

d) Suppose that $X$ has a countable base at each $x \in X$. Prove that $f : X \to Y$ is continuous if and only if

$$x = \lim_{n\to\infty} x_n \Rightarrow f(x) = \lim_{n\to\infty} f(x_n)$$

3) Let $X$ be an uncountable set of points. Let

$$\mathcal{T} = \left\{ Y \subset X \;\middle|\; X \setminus Y \text{ finite} \right\} \cup \{\emptyset\}$$

a) Prove that $\mathcal{T}$ is a topology.

b) Prove that there is no countable base at any $x \in X$.

c) Prove that $x = \lim_{n\to\infty} x_n$ for all $x \in X$ and all sequences $\{x_n\} \subset X$ obeying $x_n \neq x_m$ for all $n \neq m$.

4) Let $\langle X, \mathcal{T} \rangle$ be a topological space. Let $Y$ be a subset of $X$. Define

$$\mathcal{S} = \left\{ \mathcal{O} \cap Y \;\middle|\; \mathcal{O} \in \mathcal{T} \right\}$$

a) Prove that $\langle Y, \mathcal{S} \rangle$ is a topological space. (It is called a subspace of $\langle X, \mathcal{T} \rangle$ and $\mathcal{S}$ is called the relative topology induced by $\mathcal{T}$.)

b) Let $Y$ be an open subset of $X$. Prove that $A \subset Y$ is open in $Y$ if and only if it is open in $X$.

c) Let $Y$ be a closed subset of $X$. Prove that $A \subset Y$ is closed in $Y$ if and only if it is closed in $X$.

see over

5) Let $\mathcal{Y}$ and $\Omega_\alpha,\ \alpha \in \mathcal{I}$ be topological spaces.

a) Prove that a sequence $\{\ x_n \bigm| n \in \mathbb{N}\ \} \subset \times_{\alpha\in\mathcal{I}} \Omega_\alpha$ converges to $x$ if and only if $x(\alpha) = \lim x_n(\alpha)$ for all $\alpha \in \mathcal{I}$.

b) Prove that $f = (f_\alpha)_{\alpha\in\mathcal{I}} : \mathcal{Y} \to \times_{\alpha\in\mathcal{I}} \Omega_\alpha$ is continuous if and only if $f_\alpha : \mathcal{Y} \to \Omega_\alpha$ is continuous for all $\alpha \in \mathcal{I}$.

6) A topological space $X$ is called disconnected if there exist nonempty open sets $U, V$ such that $U \cap V = \emptyset$ and $U \cup V = X$. Otherwise $X$ is called connected. A subset of $X$ is called connected/disconnected if it is connected/disconnected as a topological space with the relative topology inherited from $X$.

a) Prove that $X$ is connected if and only if the only subsets of $X$ that are both open and closed are $\emptyset$ and $X$.

b) Prove that if $\{E_\alpha\}_{\alpha\in A}$ is a collection of connected subsets of $X$ such that $\bigcap_{\alpha\in A} E_\alpha \neq \emptyset$, then $\bigcup_{\alpha\in A} E_\alpha$ is connected.

c) Prove that if $A \subset X$ is connected, then $\bar{A}$ is connected.

d) Prove that every point $x \in X$ is contained in a unique maximal connected subset of $X$ and that this subset is closed. It is called the connected component of $x$.

The following is NOT part of Problem Set I. It will be part of Problem Set II.

1) Let $X$ and $Y$ be topological spaces.

a) Prove that if $X$ is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.

b) $X$ is called arcwise connected, if, for all $x_0, x_1 \in X$, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. Prove that every arcwise connected space is connected.

c) Let $X = \{\ (s,t) \in \mathbb{R}^2 \bigm| s \neq 0,\ t = \sin\frac{1}{s}\ \} \cup \{(0,0)\}$ with the relative topology inherited from $\mathbb{R}^2$. Prove that $X$ is connected but not arcwise connected.

2) A collection $\mathcal{F}$ of sets is said to have the *finite intersection property* if each finite subcollection of $\mathcal{F}$ has a nonempty intersection. Prove that a topological space is compact if and only if every collection of closed sets with the finite intersection property has a nonempty intersection.

A topological space is said to be **sequentially compact** if every sequence has a convergent subsequence. For metric spaces, sequential compactness is equivalent to compactness. However for general topological spaces, a compact space need not be sequentially compact (see Folland §4.4, Problem 43 or Royden §9.9, Problem 41) and conversely, a sequentially compact space need not be compact (see Folland §4.4, Problem 42). On the other hand, there is a generalization of sequential compactness (with countable "sequences" replaced by, possibly uncountable, "nets") that is equivalent to compactness. See Folland §4.3 and §4.4.

3) Let $X$ be a topological space. Prove that if $X$ is compact and first countable, meaning that it has a countable base at each point (see Problem Set I, #2c), then $X$ is sequentially compact.

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