Various Inequalities

Theorem. *Let* $< X, \Sigma, \mu >$ *be a measure space. Then*

a) (Minkowski) If $1 \le p \le \infty$*, then*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

If $1 < p < \infty$*, there is equality if and only* $\|g\|_p\, f(x) = \|f\|_q\, g(x)$ *for almost all* $x \in X$*.*

b) If $0 < p < 1$ *and* $f(x), g(x) \ge 0$ *a.e. then*

$$\|f+g\|_p \ge \|f\|_p + \|g\|_p$$

c) (Hölder) Let $1 \le p, q \le \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $f \in \mathcal{L}^p$ *and* $g \in \mathcal{L}^q$ *then* $fg \in \mathcal{L}^1$ *and*

$$\int |fg| \; d\mu \le \|f\|_p \, \|g\|_q$$

with equality if and only if there exist constants $\alpha, \beta \ge 0$*, not both zero, such that* $\alpha|f(x)|^p = \beta|g(x)|^q$ *for almost all* $x \in X$*.*

d) (Generalized Hölder) Let $1 \le r \le \infty$ *and* $1 \le p_j \le \infty$ *with* $\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r}$*. If* $f_j \in \mathcal{L}^{p_j}$ *for* $1 \le j \le n$*, then* $\prod_{j=1}^n f_j \in \mathcal{L}^r$ *and*

$$\big\| \prod_{j=1}^n f_j \big\|_r \le \prod_{j=1}^n \|f_j\|_{p_j}$$

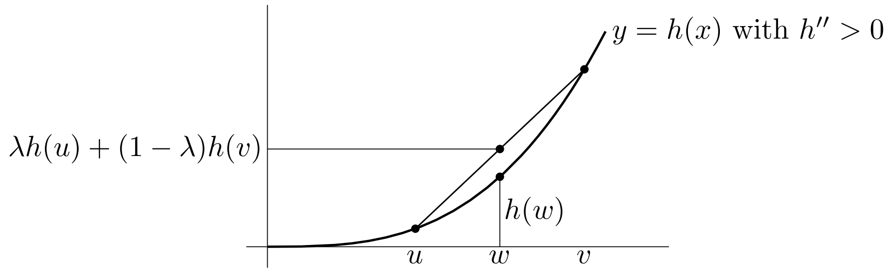
Proof of a) and b):
Reductions: Since $|f(x)| \le \|f\|_\infty$ and $|g(x)| \le \|g\|_\infty$ for almost all x, it is obvious that $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$. So we may assume that $p < \infty$. Also if $\|f\|_p = 0$ or $\|g\|_p = 0$, then $f = 0$ a.e. or $g = 0$ a.e. and $\|f+g\|_p = \|f\|_p + \|g\|_p$. So we may assume that $\|f\|_p, \|g\|_p > 0$. By replacing f by $\frac{f}{\|f\|_p + \|g\|_p}$ and g by $\frac{g}{\|f\|_p + \|g\|_p}$, we may assume that $\|f\|_p + \|g\|_p = 1$.
Concavity: Define $h(y) = y^p$. Observe that for $y > 0$

$$h''(y) = p(p-1)y^{p-2} \begin{cases} > 0 & \text{if } p > 1 \\ = 0 & \text{if } p = 1 \\ < 0 & \text{if } 0 < p < 1 \end{cases}$$

That is, h is concave up for $p > 1$, linear for $p = 1$ and concave down for $0 < p < 1$. Thus for all $u, v \ge 0$ and $0 \le \lambda \le 1$

$$h\big(\lambda u + (1-\lambda)v\big) \left\{ \begin{matrix} > & \text{if } p > 1 \\ = & \text{if } p = 1 \\ < & \text{if } p < 1 \end{matrix} \right\} \big[\lambda h(u) + (1-\lambda)h(v)\big] \tag{1}$$

For $p > 1$, there is equality if and only if $w = \lambda u + (1-\lambda)v$ equals u or v. For $0 < \lambda < 1$, this is the case if and only if $u = v$.



Proof of a): Recall that we have reduced consideration to $\|f\|_p, \|g\|_p \neq 0$, $\|f\|_p + \|g\|_p = 1$ and $1 < p < \infty$. Setting $\lambda = \|f\|_p$,

$$\begin{aligned}
\|f+g\|_p^p &= \int |f(x)+g(x)|^p \; d\mu(x) \\
&= \int \big|\lambda \tfrac{f(x)}{\|f\|_p} + (1-\lambda)\tfrac{g(x)}{\|g\|_p}\big|^p \; d\mu(x) \\
&\le \int \big[\lambda \tfrac{|f(x)|}{\|f\|_p} + (1-\lambda)\tfrac{|g(x)|}{\|g\|_p}\big]^p \; d\mu(x) \\
&\le \int \big[\lambda \tfrac{|f(x)|^p}{\|f\|_p^p} + (1-\lambda)\tfrac{|g(x)|^p}{\|g\|_p^p}\big] \; d\mu(x) \\
&= \lambda + (1-\lambda) = 1
\end{aligned}$$

by (1) with $u = \frac{|f(x)|}{\|f\|_p}$ and $v = \frac{|g(x)|}{\|g\|_p}$. For the second inequality to be an equality, we need $u = \frac{|f(x)|}{\|f\|_p} = v = \frac{|g(x)|}{\|g\|_p}$ for almost all x. For complex numbers a and b, $|a+b| = |a| + |b|$ if and only if there is an angle ϕ such that $a = e^{i\phi}|a|$ and $b = e^{i\phi}|b|$. In the real case, $|a+b| = |a| + |b|$ if and only if a and b have the same sign. Thus for the first inequality to be an equality, there must be a real valued function $\phi(x)$ such that $|f(x)| = e^{-i\phi(x)}f(x)$ and $|g(x)| = e^{-i\phi(x)}g(x)$ for almost all x. All together, if $\|f+g\|_p = \|f\|_p + \|g\|_p$, then $\frac{f(x)}{\|f\|_p} = \frac{g(x)}{\|g\|_p}$ for almost all x.

Proof of b): We are assuming that $f(x), g(x) \ge 0$ and we have again reduced consideration to $\|f\|_p, \|g\|_p \neq 0$, $\|f\|_p + \|g\|_p = 1$. With $\lambda = \|f\|_p$,

$$\begin{aligned}
\|f+g\|_p^p &= \int [f(x)+g(x)]^p \; d\mu(x) \\
&= \int \big[\lambda \tfrac{f(x)}{\|f\|_p} + (1-\lambda)\tfrac{g(x)}{\|g\|_p}\big]^p \; d\mu(x) \\
&\ge \int \big[\lambda \tfrac{f(x)^p}{\|f\|_p^p} + (1-\lambda)\tfrac{g(x)^p}{\|g\|_p^p}\big] \; d\mu(x) \\
&= \lambda + (1-\lambda) = 1
\end{aligned}$$

by (1) with $u = \frac{f(x)}{\|f\|_p}$ and $v = \frac{g(x)}{\|g\|_p}$.

Proof of c):

Reductions: Since $|f(x)g(x)| \le |g(x)|\,\|f\|_\infty$ and $|f(x)g(x)| \le |f(x)|\,\|g\|_\infty$ for almost all x, the cases $p = 1, q = \infty$ and $p = \infty, q = 1$ are obvious. So we may assume that $1 < p, q < \infty$. Also if $\|f\|_p = 0$

or $\|g\|_q = 0$, then $f = 0$ a.e. or $g = 0$ a.e. and $\|fg\|_1 = 0$. So we may assume that $\|f\|_p, \|g\|_q > 0$. By replacing f by $\frac{f}{\|f\|_p}$ and g by $\frac{g}{\|g\|_q}$, we may assume that $\|f\|_p = \|g\|_q = 1$.
Preliminaries: Define $f(c) = \frac{c^p}{p} + \frac{1}{q} - c$, for $c \geq 0$. Observe that $f'(c) = c^{p-1} - 1$ is negative for $c < 1$, zero for $c = 1$ and positive for $c > 1$. Thus f is decreasing for $0 \leq c < 1$ and increasing for $c > 1$, so that the minimum value of f is 0 and is achieved only at $c = 1$. Set, for $a, b > 0$, $c = ab^{-q/p}$. Then

$$0 \leq f(c) = \tfrac{a^p}{pb^q} + \tfrac{1}{q} - ab^{-q/p} \implies \tfrac{a^p}{p} + \tfrac{b^q}{q} \geq ab^{q-q/p} = ab \tag{2}$$

since $q\big(1 - \frac{1}{p}\big) = q\frac{1}{q} = 1$. Furthermore, there is equality if and only if $1 = c = ab^{-q/p}$ i.e. $b^q = a^p$.
Proof of c: Using (2) with $a = |f(x)|$ and $b = |g(x)|$

$$\int |f(x)|\ |g(x)|\ d\mu(x) \leq \int \Big[\tfrac{|f(x)|^p}{p} + \tfrac{|g(x)|^q}{q}\Big]\ d\mu(x) = \tfrac{1}{p}\|f\|_p + \tfrac{1}{q}\|g\|_q = \tfrac{1}{p} + \tfrac{1}{q} = 1$$

Proof of d):
First we deal with $n = 2$. By Hölder, with $f = |f_1|^r$, $g = |f_2|^r$, $p = \frac{p_1}{r}$ and $q = \frac{p_2}{r}$,

$$\begin{aligned}
\|f_1f_2\|_r^r &= \int |f_1(x)f_2(x)|^r\ d\mu(x) \leq \||f_1|^r\|_{p_1/r}\||f_2|^r\|_{p_2/r} \\
&= \Big[\int |f_1(x)|^{r(p_1/r)}\ d\mu(x)\Big]^{r/p_1}\Big[\int |f_2(x)|^{r(p_2/r)}\ d\mu(x)\Big]^{r/p_2} \\
&= \|f_1\|_{p_1}^r\|f_2\|_{p_2}^r
\end{aligned}$$

Now we proceed by induction. Once the inequality has been established for $n-1$, we apply the $n = 2$ inequality, with f_2 replaced by $\prod_{j=1}^n f_j$ and p_2 replaced by $r' = \big[\sum_{j=2}^n \frac{1}{p_j}\big]^{-1}$.

$$\Big\|\prod_{j=1}^n f_j\Big\|_r \leq \|f_1\|_{p_1}\Big\|\prod_{j=2}^n f_j\Big\|_{r'}$$

Now just apply the $n-1$ inequality. ■