MATH 420 Problem Set 8 Solutions

1. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f : X \to [0, \infty)\) a measurable function. Define, for each \(E \in \mathcal{M}, \lambda(E) = \int_E f \, d\mu\). Prove that \(\lambda\) is a measure on \(\mathcal{M}\) and that \(\int g \, d\mu = \int g \, d\lambda\) for all measurable functions \(g : X \to [0, \infty)\).

Solution. Proof that \(\lambda\) is a measure: That \(\lambda(E) = \int_E f \, d\mu \geq 0\) is obvious because \(f(x) \geq 0\) for all \(x \in X\). That \(\lambda(\emptyset) = \int_{\emptyset} f \, d\mu = \int f \, d\mu = 0\) is also obvious. If \(\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}\) is a countable collection of disjoint subsets of \(X\) and \(E = \bigcup_{n=1}^{\infty} E_n\), then

\[
\lambda(E) = \int_E f \, d\mu = \int \sum_{n=1}^{\infty} \chi_{E_n} f \, d\mu = \sum_{n=1}^{\infty} \int \chi_{E_n} f \, d\mu = \sum_{n=1}^{\infty} \lambda(E_n)
\]

by the monotone convergence theorem. So we have countable additivity.

Proof that \(\int g f \, d\mu = \int g \, d\lambda\): If \(g\) is the simple function \(\sum_{j=1}^{n} a_j \chi_{E_j}\) for some \(\{a_1, \ldots, a_n\} \subset \mathbb{R}\) and \(\{E_1, \ldots, E_n\} \subset \mathcal{M}\), then

\[
\int g f \, d\mu = \int \left( \sum_{j=1}^{n} a_j \chi_{E_j} \right) f \, d\mu = \sum_{j=1}^{n} a_n \int \chi_{E_j} f \, d\mu = \sum_{j=1}^{n} a_n \lambda(E_j) = \int g \, d\lambda
\]

For general measurable \(g : X \to [0, \infty)\), let, for each \(m, n \in \mathbb{N}\), \(E_{m,n} = \{ x \in X \mid \frac{m-1}{m} \leq g(x) < \frac{m}{m} \}\) and, for each \(n \in \mathbb{N}\), \(g_n = \sum_{m=1}^{\infty} \frac{m-1}{m} \chi_{E_{m,n}}\). Then \(\{g_n\}\) is a sequence of simple, measurable functions that increase pointwise to \(g\). Hence, by the monotone convergence theorem (twice),

\[
\int g f \, d\mu = \int \lim_{n \to \infty} g_n f \, d\mu = \lim_{n \to \infty} \int g_n f \, d\mu = \lim_{n \to \infty} \int g_n \, d\lambda = \int \lim_{n \to \infty} g_n \, d\lambda = \int g \, d\lambda
\]

2. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f_n : X \to [0, \infty)\) be a sequence of measurable functions that decreases to the function \(f\). Prove that if \(\int f_1 \, d\mu < \infty\), then \(\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu\). Is the implication still valid when \(\int f_1 \, d\mu = \infty\)?

Solution. Set \(g = f_1 - f\) and, for each \(n \in \mathbb{N}\), \(g_n = f_1 - f_n\). Then \(\{g_n\}\) is a sequence of nonnegative, measurable functions that increases pointwise to \(g\). Hence, by the monotone convergence theorem,

\[
\int f_1 \, d\mu - \int f \, d\mu = \int g \, d\mu = \int \lim_{n \to \infty} g_n \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu = \lim_{n \to \infty} \left( \int f_1 \, d\mu - \int f_n \, d\mu \right)
\]

\[
= \int f_1 \, d\mu - \lim_{n \to \infty} \int f_n \, d\mu
\]

This gives the desired result, when \(\int f_1 \, d\mu < \infty\). If \(\int f_1 \, d\mu = \infty\) then the conclusion \(\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu\) need not be true. For example, if \(X = \mathbb{R}\), \(\mu\) is the Lebesgue measure and \(f_n\) is the characteristic function of \([n, \infty)\), then \(f = 0\), \(\int f \, d\mu = 0\), but \(\int f_n \, d\mu = \infty\) for all \(n \in \mathbb{N}\).

3. Prove the following variant of Fatou’s lemma:

Let \((X, \mathcal{M}, \mu)\) be a measure space and \(E \in \mathcal{M}\). Let \(g \in L^1(X)\) be nonnegative. Assume that, for each \(n \in \mathbb{N}\), \(f_n : X \to \mathbb{R}\) is measurable and \(f_n(x) \geq -g(x)\) for all \(x \in E\). Then

\[
\int_E \liminf_{n \to \infty} f_n(x) \, d\mu(x) \leq \liminf_{n \to \infty} \int_E f_n(x) \, d\mu(x)
\]

What is the analog of Fatou’s lemma for nonpositive functions?
Note: In class, we defined \( \int f(x) \, d\mu(x) \) only for \( f \in L^1(X, \mu) \) and for \( f \geq 0 \) measurable. The \( f_n \)'s in this problem need not be \( L^1 \) and need not nonnegative. Part of this problem is to extend the definition of \( \int f(x) \, d\mu(x) \) to a class of functions that include the \( f_n \)'s.

Solution. To this point, we have only defined \( \int f(x) \, d\mu(x) \) for \( f \in L^1(X, \mu) \) and for \( f \geq 0 \) measurable. We now extend the definition to any measurable function \( f : X \to \mathbb{R} \) for which at least one of \( \max\{f, 0\} \), \( \max\{-f, 0\} \) is in \( L^1(X, \mu) \) by

\[
\int f(x) \, d\mu(x) = \begin{cases} 
+\infty & \text{if } \max\{f, 0\} \notin L^1, \max\{-f, 0\} \in L^1 \\
-\infty & \text{if } \max\{f, 0\} \in L^1, \max\{-f, 0\} \notin L^1 \\
\max\{f, 0\} \, d\mu - \max\{-f, 0\} \, d\mu & \text{if } \max\{f, 0\} \in L^1, \max\{-f, 0\} \in L^1
\end{cases}
\]

The \( f_n \)'s of this question are in this class for every \( n \in \mathbb{N} \). Furthermore, if \( f \) is in this class, \( g \in L^1 \) and \( a, b \in \mathbb{R} \), then \( af + bg \) is in this class and \( \int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu \) (under the convention that \( 0 \cdot \infty = 0 \)).

Define \( h_n(x) = f_n(x) + g(x) \). Then \( h_n \geq 0 \) and by (the original) Fatou’s lemma

\[
\int_E \liminf_{n \to \infty} f_n(x) \, d\mu(x) = \int_E \left\{ \liminf_{n \to \infty} h_n(x) - g(x) \right\} \, d\mu(x)
\]

\[
= \int_E \liminf_{n \to \infty} h_n(x) \, d\mu(x) - \int_E g(x) \, d\mu(x) \quad \text{since } g \in L^1
\]

\[
\leq \liminf_{n \to \infty} \int_E h_n(x) \, d\mu(x) - \int_E g(x) \, d\mu(x)
\]

\[
= \liminf_{n \to \infty} \int_E [h_n(x) - g(x)] \, d\mu(x)
\]

\[
= \liminf_{n \to \infty} \int_E f_n(x) \, d\mu(x)
\]

Now for the analog of Fatou’s lemma for nonpositive functions. If \( f_n : X \to \mathbb{R} \) is measurable and nonpositive, then, setting \( h_n = -f_n \)

\[
\int_E \limsup_{n \to \infty} f_n(x) \, d\mu(x) = \int_E \limsup_{n \to \infty} -h_n(x) \, d\mu(x)
\]

\[
= - \int_E \liminf_{n \to \infty} h_n(x) \, d\mu(x)
\]

\[
\geq - \liminf_{n \to \infty} \int_E h_n(x) \, d\mu(x) \quad \text{since } a \leq b \Rightarrow -a \geq -b
\]

\[
= \limsup_{n \to \infty} \int_E -h_n(x) \, d\mu(x)
\]

\[
= \limsup_{n \to \infty} \int_E f_n(x) \, d\mu(x)
\]

So the analog of Fatou’s lemma for nonpositive functions is

Lemma. Let \((X, \mathcal{M}, \mu)\) be a measure space and \( E \in \mathcal{M} \). Let, for each \( n \in \mathbb{N} \), \( f_n : X \to \mathbb{R} \) be measurable and nonpositive. Then

\[
\int_E \limsup_{n \to \infty} f_n(x) \, d\mu(x) \geq \limsup_{n \to \infty} \int_E f_n(x) \, d\mu(x)
\]
4. Compute the following limits and justify the calculations.

(a) \( \lim_{n \to \infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin \frac{x}{n} \, dx \)

(b) \( \lim_{n \to \infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx \)

(c) \( \lim_{n \to \infty} \int_0^\infty \frac{1}{1+nx^2} n \sin \frac{x}{n} \, dx \)

(d) \( \lim_{n \to \infty} \int_a^\infty \frac{n}{1+nx^2} \, dx \) (The answer depends on whether \( a > 0 \), \( a = 0 \) or \( a < 0 \). How does this accord with the various convergence theorems?)

**Solution.**

(a) By the binomial expansion, if \( x \geq 0 \) and \( n \geq 2 \), then

\[
(1 + \frac{x}{n})^n \geq 1 + nx + \binom{n}{2} \left( \frac{x}{n} \right)^2 = 1 + x + \frac{n-1}{2n} x^2 \geq 1 + x + \frac{1}{4} x^2
\]

Since the integrand is bounded in absolute value by the \( L^1 \) function \( \frac{1}{1+x^2+x^4/4} \) and converges pointwise to zero as \( n \to \infty \) (since \( \sin \frac{x}{n} \) converges to zero as \( n \to \infty \) for every fixed \( x \) and the other factor is uniformly bounded), the integral converges to zero by the dominated convergence theorem.

(b) By the binomial expansion, if \( x \geq 0 \) and \( n \geq 2 \), then

\[
(1 + x^2)^n \geq 1 + nx^2 + \binom{n}{2} x^4 = 1 + nx^2 + \frac{n(n-1)}{2} x^4
\]

so that the integrand

\[
(1 + nx^2)(1 + x^2)^{-n} \leq \frac{1 + nx^2}{1 + nx^2 + \frac{n(n-1)}{2} x^4}
\]

is bounded, for \( n \geq 2 \), by the \( L^1 \) function 1 and converges pointwise to zero as \( n \to \infty \), except at \( x = 0 \). So the integral converges to zero by the dominated convergence theorem.

(c) Recall that \( |\sin y| \leq |y| \) for all \( y \in \mathbb{R} \). (This obvious for \( |y| \geq 1 \) since \( |\sin y| \leq 1 \). For \( 0 \leq y \leq 1 \) use the Taylor series expansion for \( \sin y \) and the alternating series test. For \( y < 0 \), use \( |\sin y| = |\sin |y||. \)

Consequently \( n \sin \frac{x}{n} \leq x \) for all \( x \geq 0 \) and \( n \in \mathbb{N} \). So the integrand \( \frac{1}{1+x^2} \sin \frac{x}{n} \) is bounded in magnitude by the \( L^1 \) function \( \frac{1}{1+x^2} \) and converges pointwise to \( \frac{1}{1+x^2} \) as \( n \to \infty \) (since \( \lim_{y \to 0} \sin y / y = 1 \)).

So, by the dominated convergence theorem, the integral converges to

\[
\int_0^\infty \frac{1}{1+x^2} \, dx = \frac{\pi}{2}
\]

(d) Making the change of variables \( y = nx \)

\[
\int_a^\infty \frac{n}{1+n^2} \, dx = \int_{na}^\infty \frac{1}{1+y^2} \, dy = -\arctan(na) \overset{n \to \infty}{\longrightarrow} \frac{\pi}{2} \begin{cases} 
\frac{\pi}{2} & \text{if } a = 0 \\
\frac{\pi}{4} & \text{if } a < 0 \\
0 & \text{if } a > 0
\end{cases}
\]

Before making the change of variables, the integrand converges pointwise to zero as \( n \to \infty \), except at \( x = 0 \). For \( a > 0 \), we may apply the dominated convergence theorem with \( g(x) = \frac{1}{2x^2} \), since \( \frac{n}{1+n^2} \, dx \leq \frac{n}{n^2x^2} \leq \frac{1}{2x^2} \). But for \( a \leq 0 \), the integrand is not bounded by a fixed \( L^1 \) function for all \( n \) (for example, if \( n = \frac{1}{x} \), then \( \frac{n}{1+n^2x^2} = \frac{1}{2x^2} \)), so we may not apply the usual convergence theorems.

5. Let \( f \in L^1(\mathbb{R}, m) \), with \( m \) being Lebesgue measure, and set \( F(x) = \int_x^\infty f(t) \, dm(t) \). Prove that \( F \) is continuous on \( \mathbb{R} \).

**Solution.** Suppose that \( f \) is not continuous on \( \mathbb{R} \). Then there are \( x \in \mathbb{R} \), \( \varepsilon > 0 \) and a sequence of points \( \{x_n\} \in \mathbb{N} \) in \( \mathbb{R} \) that converge to \( x \) such that \(|F(x) - F(x_n)| > \varepsilon \) for all \( n \in \mathbb{N} \). Set

\[
h_n(t) = \begin{cases} 
f(t) & \text{if } t \leq x_n \\
0 & \text{otherwise}
\end{cases} \quad h(t) = \begin{cases} 
f(t) & \text{if } t \leq x \\
0 & \text{otherwise}
\end{cases}
\]

Then \(|h_n(t)|, |h(t)| \leq |f(t)| \in L^1 \) and \( \{h_n\} \) converges pointwise almost everywhere (in fact everywhere, except possibly at \( x \)) to \( h(x) \). So, by the dominated convergence theorem,

\[
\lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} \int h_n(t) \, dt = \int h(t) \, dt = F(x)
\]

which contradicts \(|F(x) - F(x_n)| \geq \varepsilon \).