1. Let \( m \) be the Lebesgue measure on \( \mathbb{R} \), \( m^* \) be the corresponding outer measure and \( \mathcal{L} \) be the collection of Lebesgue measurable sets. Define, for any \( E \subseteq \mathbb{R} \) and any \( t \in \mathbb{R} \), the sets \( E + t = \{ x + t \mid x \in E \} \) and \( tE = \{ tx \mid x \in E \} \).

(a) Prove that if \( E \subseteq \mathbb{R} \) and \( t \in \mathbb{R} \), then \( m^*(E + t) = m^*(E) \) and \( m^*(tE) = |t| m^*(E) \).

(b) Prove that if \( E \in \mathcal{L} \) and \( t \in \mathbb{R} \), then \( E + t \in \mathcal{L} \), \( tE \in \mathcal{L} \), \( m(E + t) = m(E) \) and \( m(tE) = |t| m(E) \).

**Solution.** (a) Denote

\[
M(E) = \left\{ \sum_j (b_j - a_j) \mid E \subseteq \bigcup (a_j, b_j) \right\}
\]

The countable collection \( \{(a_j, b_j)\} \) of open intervals covers \( E \) if and only if the countable collection \( \{(a_j + t, b_j + t)\} \) of open intervals covers \( E + t \). Since \( \sum_j [(b_j + t) - (a_j + t)] = \sum_j [b_j - a_j] \), we have that \( M(E) = M(E + t) \) and \( m^*(E) = \inf M(E) = \inf M(E + t) = m^*(E + t) \).

- Let \( t > 0 \). The countable collection \( \{(a_j, b_j)\} \) of open intervals covers \( E \) if and only if the countable collection \( \{(ta_j, tb_j)\} \) of open intervals covers \( tE \). Since \( \sum_j (tb_j - ta_j) = t \sum_j [b_j - a_j] \), we have that \( M(tE) = tM(E) \) and \( m^*(tE) = \inf tM(E) = \inf tm(E) = tm^*(E) \).

- Let \( t < 0 \). The countable collection \( \{(a_j, b_j)\} \) of open intervals covers \( E \) if and only if the countable collection \( \{(tb_j, ta_j)\} \) of open intervals covers \( tE \). Since \( \sum_j (ta_j - tb_j) = |t| \sum_j [b_j - a_j] \), we have that \( M(tE) = |t|M(E) \) and \( m^*(tE) = \inf |t|M(E) = \inf |t|m(E) = |t|m^*(E) \).

- Finally, if \( t = 0 \), then,

\[
tE = \begin{cases} 0 & \text{if } E = \emptyset \\ \{0\} & \text{if } E \neq \emptyset \end{cases}
\]

and in both cases \( tE \in \mathcal{L} \) with \( m^*(tE) = 0 = |t|m^*(E) \).

(b) I’ll give the proof for \( E + t \). The other cases are virtually identical. Let \( F \subseteq \mathbb{R} \) and set \( F' = F - t \). By part (a),

\[
m^*(F) = m^*(F' + t) = m^*(F')
\]

\[
m^*(F \cap (E + t)) = m^*((F' \cap t) \cap (E + t)) = m^*((F' \cap E) + t) = m^*(F' \cap E)
\]

\[
m^*(F \cap (E + t)^c) = m^*((F' + t) \cap (E^c + t)) = m^*((F' \cap E^c) + t) = m^*(F' \cap E^c)
\]

Hence

\[
m^*(F) = m^*(F \cap (E + t)) + m^*(F \cap (E + t)^c) \iff m^*(F') = m^*(F' \cap E) + m^*(F' \cap E^c)
\]

So \( E + t \in \mathcal{L} \iff E \in \mathcal{L} \). By part (a), if \( E, E + t \in \mathcal{L} \), then \( m(E) = m^*(E) = m^*(E + t) = m(E + t) \).

2. Let \( X \) and \( Y \) be nonempty sets and let \( \mathcal{N} \) be a \( \sigma \)-algebra on \( Y \) and let \( f : X \to Y \). Prove that \( \mathcal{M} = \{ f^{-1}(N) \mid N \in \mathcal{N} \} \) is a \( \sigma \)-algebra and is the smallest \( \sigma \)-algebra on \( X \) with respect to which \( f \) is measurable. It is sometimes called the \( \sigma \)-algebra generated by \( f \).

**Solution.** Proof that \( \mathcal{M} \) is a \( \sigma \)-algebra: If \( A \in \mathcal{M} \), there is an \( N \in \mathcal{N} \) such that \( A = f^{-1}(N) \). Then \( A^c = f^{-1}(N^c) \in \mathcal{M} \), so \( \mathcal{M} \) is closed under taking complements. If, for each \( n \in \mathbb{N} \), \( A_n \in \mathcal{M} \), then there is, for each \( n \in \mathbb{N} \), an \( N_n \in \mathcal{N} \) such that \( A_n = f^{-1}(N_n) \). Then \( \bigcup_n A_n = \bigcup_n f^{-1}(N_n) = f^{-1}(\bigcup_n N_n) \in \mathcal{M} \), so \( \mathcal{M} \) is closed under countable unions. So \( \mathcal{M} \) is a \( \sigma \)-algebra.

Proof of minimality: Let \( \mathcal{M}' \) be a \( \sigma \)-algebra of subsets of \( X \) and assume that \( f \) is \( (\mathcal{M}', \mathcal{N}) \)-measurable. Then, for each \( N \in \mathcal{N} \), \( f^{-1}(N) \in \mathcal{M}' \). Hence \( \mathcal{M} \subseteq \mathcal{M}' \).
3. Let \( \{X, \mathcal{M}\} \) be a measure space and \( f : X \to \mathbb{R} \). Prove that \( f \) is measurable if and only if \( f^{-1}( (q, \infty) ) \in \mathcal{M} \) for all \( q \in \mathbb{Q} \).

**Solution.** It suffices to prove that \( \{ (q, \infty) \mid q \in \mathbb{Q} \} \) generates \( \mathcal{B}_{\mathbb{R}} \). Since \( \{ (a, \infty) \mid a \in \mathbb{R} \} \) generates \( \mathcal{B}_{\mathbb{R}} \), the desired conclusion follows from the observation that, for each \( a \in \mathbb{R} \), there is a sequence \( \{q_n\}_{n \in \mathbb{N}} \) of rational numbers that decrease to \( a \) so that

\[
(a, \infty) = \bigcup_{n=1}^{\infty} (q_n, \infty)
\]

4. Let \( \{X, \mathcal{M}\} \) be a measure space. Find sufficient conditions on \( \{X, \mathcal{M}\} \) such that there exists a bounded family of measurable functions \( \{f_\alpha : X \to \mathbb{R}\}_{\alpha \in \mathcal{I}} \) whose supremum is not measurable. (\( \mathcal{I} \) will be uncountable.)

**Solution.** It suffices that \( \mathcal{M} \neq \mathcal{P}(X) \) and that there exists a subset \( E \) of \( X \) that is not measurable but that is an (uncountable) disjoint union \( \bigcup_{\alpha \in \mathcal{I}} E_\alpha \) of measurable sets. (This will be the case if, for example, every single point subset of \( X \) is measurable but \( \mathcal{M} \neq \mathcal{P}(X) \).) Then, the characteristic function \( \chi_{E_\alpha} \) is measurable for each \( \alpha \in \mathcal{I} \) but

\[
\chi_E = \sup_{\alpha \in \mathcal{I}} \chi_{E_\alpha}
\]

is not measurable.

5. Prove that if \( f : \mathbb{R} \to \mathbb{R} \) is monotone, then it is Borel measurable.

**Solution.** We may assume, without loss of generality, that \( f \) is nondecreasing. (Otherwise, replace \( f \) by \( -f \).) Let \( a \in \mathbb{R} \). Set

\[
M_a = \inf \{ x \in \mathbb{R} \mid f(x) > a \}
\]

(with \( M_a = -\infty \) when \( f(x) > a \) for all \( x \in \mathbb{R} \) and \( M_a = \infty \) when \( f(x) \leq a \) for all \( x \in \mathbb{R} \)). If, for some \( x \in \mathbb{R} \), \( f(x) > a \), then \( f(y) > a \) for all \( y \geq x \). Hence \( f^{-1}( (a, \infty) ) \) is either \( \emptyset \) (if \( M_a = \infty \)) or \( \mathbb{R} \) (if \( M_a = -\infty \)) or \( (M_a, \infty) \) or \( [M_a, \infty) \). These are all Borel sets, so \( f^{-1}( (a, \infty) ) \) is a Borel set. By a Lemma proven in class, this implies that \( f \) is Borel measurable.

6. Let \( \{X, \mathcal{M}\} \) be a measure space and \( \{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}} \) a sequence of measurable functions. Prove that

\[
\{ x \in X \mid \lim_{n \to \infty} f_n(x) \text{ exists} \} \in \mathcal{M}.
\]

**Solution.** Recall that \( \lim_{n \to \infty} f_n(x) \) exists if and only if the sequence \( \{f_n(x)\}_{n \in \mathbb{N}} \) of real numbers is Cauchy. That is

\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } m, n \geq N \Rightarrow |f_m(x) - f_n(x)| < \varepsilon
\]

Define, for each \( \varepsilon > 0 \) and \( n, m \in \mathbb{N} \)

\[
E_{\varepsilon, m, n} = \{ x \in X \mid -\varepsilon < f_m(x) - f_n(x) < \varepsilon \}
\]

As \( f_m - f_n \) is measurable, we have that \( E_{\varepsilon, m, n} \in \mathcal{M} \). As \( \mathcal{M} \) is a \( \sigma \)-algebra,

\[
E_{\varepsilon, N} = \{ x \in X \mid n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon \} = \bigcap_{n, m \geq N} E_{\varepsilon, m, n} \in \mathcal{M}
\]

\[
E_{\varepsilon} = \{ x \in X \mid \exists N \in \mathbb{N} \text{ such that } n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon \} = \bigcup_{N \in \mathbb{N}} E_{\varepsilon, N} \in \mathcal{M}
\]

\[
E = \{ x \in X \mid \lim_{n \to \infty} f_n(x) \text{ exists} \} = \bigcap_{p \in \mathbb{N}} E_{2^{-p}} \in \mathcal{M}
\]
7. Let $X$ be a nonempty set and $\mathcal{M}$ a $\sigma$-algebra of subsets of $X$. A function $f : X \to \mathbb{R}$ is said to be measurable on $A \in \mathcal{M}$ if $f^{-1}(B) \cap A \in \mathcal{M}$ for all Borel sets $B$. Equivalently, $f$ is measurable on $A$ if the restriction of $f$ to $A$ is $\mathcal{M}_A$-measurable, where $\mathcal{M}_A = \{ E \cap A \mid E \in \mathcal{M} \}$. Let $A,B \in \mathcal{M}$ with $X = A \cup B$. Prove that $f : X \to \mathbb{R}$ is measurable if and only if $f$ is measurable on $A$ and on $B$.

Solution. Let $C \in \mathcal{B}_{\mathbb{R}}$.
- If $f$ is measurable on $A$ and on $B$, then
  
  $$f^{-1}(C) \cap A, f^{-1}(C) \cap B \in \mathcal{M} \Rightarrow f^{-1}(C) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B) \in \mathcal{M}$$

  because $\mathcal{M}$ is a $\sigma$-algebra.
- Conversely, if $f$ is measurable, then $f^{-1}(C) \in \mathcal{M}$ and, since $A,B \in \mathcal{M}$ and $\mathcal{M}$ is a $\sigma$-algebra, $f^{-1}(C) \cap A \in \mathcal{M}$ and $f^{-1}(C) \cap B \in \mathcal{M}$, so that $f$ is measurable on $A$ and on $B$.

8. Let $X$ be a nonempty set and $\mathcal{M}$ a $\sigma$-algebra of subsets of $X$. Define $\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$ and $\mathcal{B}_{\mathbb{R}} = \{ E \subseteq \mathbb{R} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \}$.

(a) Prove that $\mathcal{B}_{\mathbb{R}}$ is a $\sigma$-algebra.
(b) Prove that $f : X \to \mathbb{R}$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable if and only if $f^{-1}(\{ \infty \}) \in \mathcal{M}$, $f^{-1}(\{-\infty\}) \in \mathcal{M}$ and $f$ is measurable on $f^{-1}(\mathbb{R})$.

Solution. (a) If $E \in \mathcal{B}_{\mathbb{R}}$, then $(\mathbb{R} \setminus E) \cap \mathbb{R} = \mathbb{R} \setminus (E \cap \mathbb{R}) \in \mathcal{B}_{\mathbb{R}}$, so that $\mathbb{R} \setminus E \in \mathcal{B}_{\mathbb{R}}$. If, for each $n \in \mathbb{N}$, $E_n \in \mathcal{B}_{\mathbb{R}}$, then $(\cup_n E_n) \cap \mathbb{R} = \cup_n (E_n \cap \mathbb{R}) \in \mathcal{B}_{\mathbb{R}}$, so that $\cup_n E_n \in \mathcal{B}_{\mathbb{R}}$.

(b) Denote $X_\infty = f^{-1}(\{ \infty \})$, $X_{-\infty} = f^{-1}(\{-\infty\})$ and $X_\mathbb{R} = f^{-1}(\mathbb{R})$. These three sets are disjoint and have union $X$.

- If $f$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable, then, as $\{ \infty \}$, $\{-\infty\}$ and $\mathbb{R}$ are all in $\mathcal{B}_{\mathbb{R}}$, we have that $X_\infty$, $X_{-\infty}$ and $X_\mathbb{R}$ are all in $\mathcal{M}$. Furthermore, if $C \in \mathcal{B}_{\mathbb{R}}$, then $f^{-1}(C) \cap X_\mathbb{R} = f^{-1}(C) \cap f^{-1}(\mathbb{R}) = f^{-1}(C \cap \mathbb{R}) \in \mathcal{M}$ so that $f$ is measurable on $X_\mathbb{R}$.

- Conversely, suppose that $X_\infty, X_{-\infty} \in \mathcal{M}$ and that $f$ is measurable on $X_\mathbb{R}$. Let $C \in \mathcal{B}_{\mathbb{R}}$. Then

  $$f^{-1}(C) = f^{-1}(C \cap \{ \infty \}) \cup f^{-1}(C \cap \{-\infty\}) \cup f^{-1}(C \cap \mathbb{R})$$

  $$= (X_\infty \text{ or } \emptyset) \cup (X_{-\infty} \text{ or } \emptyset) \cup (f^{-1}(C) \cap X_\mathbb{R}) \in \mathcal{M}$$

  so that $f$ is measurable.

9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that the following implications are true if and only if $\mu$ is complete.

(a) Let $f,g : X \to \mathbb{R}$. If $f$ is measurable and $f = g$ $\mu$-a.e., then $g$ is measurable.

(b) Let $f : X \to \mathbb{R}$ and $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. If $f_n$ is measurable for all $n \in \mathbb{N}$ and $\{f_n\}$ converges $\mu$-a.e. to $f$, then $f$ is measurable.

Solution.

- First suppose that $\mu$ is complete.

  (a) Let $E \in \mathcal{M}$ be such that $\mu(E) = 0$ and $f(x) = g(x)$ for all $x \in E^c$. Then, for all $a \in \mathbb{R}$

  $$g^{-1}((a, \infty)) = \{ x \in X \mid g(x) > a \}$$

  $$= \{ x \in E^c \mid g(x) > a \} \cup \{ x \in E \mid g(x) > a \}$$

  $$= \{ x \in E^c \mid f(x) > a \} \cup \{ x \in E \mid g(x) > a \}$$

  $$= (E^c \cap f^{-1}((a, \infty))) \cup \{ x \in E \mid g(x) > a \} \in \mathcal{M}$$

  so $g$ is measurable.
(b) Let $E \in \mathcal{M}$ be such that $\mu(E) = 0$ and $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in E^c$. Define

$$\phi_n(x) = \begin{cases} f_n(x) & \text{if } x \in E^c \\ 0 & \text{if } x \in E \end{cases} \quad \phi(x) = \begin{cases} f(x) & \text{if } x \in E^c \\ 0 & \text{if } x \in E \end{cases}$$

By part (a), $\phi_n$ is measurable for all $n \in \mathbb{N}$ (since $f_n$ is measurable and $f_n = \phi_n$ a.e. for all $n \in \mathbb{N}$). Since $\phi(x) = \lim_{n \to \infty} \phi_n(x)$ for all $x \in X$, $\phi$ is measurable. By part (a), $f$ is measurable (since $\phi$ is measurable and $f = \phi$ a.e.).

• **Now suppose that $\mu$ is not complete.** Then there is an $N \in \mathcal{M}$ with $\mu(N) = 0$ and an $E \subset N$ with $E \notin \mathcal{M}$. Then $f = 0$, $g = \chi_E$ provides a counterexample for the implication of part (a) since $g^{-1}(\{1\}) = E \notin \mathcal{M}$ ensures that $g$ is not measurable. And $f_n = 0$ for all $n \in \mathbb{N}$ and $f = \chi_E$ provides a counterexample for the implication of part (b).