MATH 420 Problem Set 5 Solutions

1. Let \( X \) be a nonempty set and \( \mu^* \) an outer measure on \( X \).
   (a) Prove that if \( N \subset X \) obeys \( \mu^*(N) = 0 \), then \( N \) is measurable.
   (b) Prove that if \( A, B \subset X \) and \( A \) is measurable and \( \mu^*(A \Delta B) = 0 \), then \( B \) is measurable.
   **Solution.** (a) Let \( E \subset X \). By subadditivity, \( \mu^*(E) \leq \mu^*(E \cap N) + \mu^*(E \cap N^c) \). Since \( E \cap N \subset N \) and \( E \cap N^c \subset E \), we have \( \mu^*(E \cap N) \leq \mu^*(N) = 0 \) and \( \mu^*(E \cap N^c) \leq \mu^*(E) \) and hence \( \mu^*(E \cap N) + \mu^*(E \cap N^c) \leq \mu^*(E) \). So \( \mu^*(E) = \mu^*(E \cap N) + \mu^*(E \cap N^c) \) for all \( E \subset X \) and \( N \) is measurable.
   (b) As \( A \cap B^c, B \cap A^c \subset A \Delta B \), we have \( \mu^*(A \cap B^c) = \mu^*(B \cap A^c) = 0 \). So, by part (a), \( A \cap B^c \) and \( B \cap A^c \) are measurable. Hence so are \( (A \cap B^c)^c = A^c \cup B \) and \( B = (B \cap A) \cup (B \cap A^c) = (A \cap (A \cap B^c)^c) \cup (B \cap A^c) \) since the collection of measurable sets is a \( \sigma \)-algebra.

2. Let \( 0 \leq \alpha \leq 1 \). Find a Lebesgue measurable set \( F_\alpha \subset [0,1] \) that is dense in \([0,1]\) and has \( m(F_\alpha) = \alpha \).
   **Solution.** \( F_\alpha = \left( [0,\alpha] \cap \mathbb{R} \setminus \{Q\} \right) \cup \left( [\alpha,1] \cap \{Q\} \right) \). It is dense because both \( \{Q\} \) and \( \mathbb{R} \setminus \{Q\} \) are dense in \( \mathbb{R} \).
   It is measurable because
   \( \{Q\} \) is measurable (it has measure zero)
   \([0,\alpha] \) and \([\alpha,1] \) are measurable (they are both Borel)
   \( \mathcal{M} \) is a \( \sigma \)-algebra
   It has measure \( \alpha \) because \( m([\alpha,1] \cap \{Q\}) \leq m(\{Q\}) = 0 \) and
   \[ m([0,\alpha] \cap \mathbb{R} \setminus \{Q\}) = m([0,\alpha]) - m([0,\alpha] \cap \{Q\}) = m([0,\alpha]) = \alpha \]

3. Find \( F \) such that \( \mu = \mu_F \) (on \( B_\mathbb{R} \)), if \( \mu \) is the measure given by:
   (a) \( \mu(E) = \mu_G(E) + \mu_H(E) \) (where \( G, H \) are fixed functions)
   (b) \( \mu(E) = \mu_G(E \cap (u,v)) \) (where \( u < v \) are fixed constants)
   **Solution.** (a) For all \( a < b \),
   \[
   \mu((a,b]) = \mu_G((a,b]) + \mu_H((a,b])
   \]
   \[= G(b) - G(a) + H(b) - H(a)\]
   \[= F(b) - F(a) \text{ where } F(x) = G(x) + H(x)\]
   \[= \mu_F((a,b])\]
   So \( F(x) = G(x) + H(x) \) does the job.
   (b) For all \( a < b \),
   \[
   \mu((a,b]) = \mu_G((a,b] \cap (u,v]) = \begin{cases} 
   \mu_G(\{\max\{a,u\}, \min\{b,v\}\}) & \text{if } b > u \text{ and } a < v \\
   \mu_G(\emptyset) & \text{if } b \leq u \text{ or } a \geq v 
   \end{cases}
   \]
   \[= \begin{cases} 
   G(\min\{b,v\}) - G(\max\{a,u\}) & \text{if } b > u \text{ and } a < v \\
   0 & \text{if } b \leq u \text{ or } a \geq v 
   \end{cases}
   \]
   \[= \begin{cases} 
   G(b) - G(u) & \text{if } u < b \leq v \text{ and } a \leq u \\
   G(b) - G(a) & \text{if } u < b \leq v \text{ and } a > u \\
   G(v) - G(u) & \text{if } v < b \text{ and } a \leq u \\
   G(v) - G(a) & \text{if } v < b \text{ and } u < a < v 
   \end{cases}
   \]
   \[= F(b) - F(a) \text{ where } F(x) = \begin{cases} 
   G(x) & \text{if } x \geq v \\
   G(u) & \text{if } x < u \\
   G(x) & \text{if } u \leq x \leq v \\
   G(u) & \text{if } x \leq u 
   \end{cases}
   \]
   \[= \mu_F((a,b]) \]
   So this \( \uparrow \uparrow F \) does the job.
4. Let \( \mu \) be a Lebesgue-Stieltjes measure on \( \mathbb{R} \) and \( M_\mu \), the set of \( \mu \)-measurable sets.

(a) Prove that if \( E \in M_\mu \), then
\[
\mu(E) = \sup \{ \mu(K) \mid K \subset E, \ K \text{ compact} \}
\]

(b) Let \( E \subset \mathbb{R} \). Prove that \( E \) is \( \mu \)-measurable if and only if \( E = H \cup N \) where \( H \) is a countable union of compact sets and \( \mu^*(N) = 0 \).

**Solution.**

(a) Call the right hand side \( \nu(E) \). That \( \nu(E) \leq \mu(E) \) is obvious because \( \mu(K) \leq \mu(E) \) for all measurable \( K \subset E \).

**Proof that** \( \mu(E) \leq \nu(E) \) **when** \( E \) **is compact:** This is obvious because we may choose \( K = E \).

**Proof that** \( \mu(E) \leq \nu(E) \) **when** \( E \) **is bounded but not closed:** Let \( \epsilon > 0 \). We proved in class that there is an open set \( O \) such that \( O \supset \bar{E} \setminus E \) and \( \nu(O) \leq \mu(\bar{E} \setminus E) + \epsilon \). Then \( K = \bar{E} \setminus O \) is compact and is contained in \( E \) since \( \bar{E} \cap O^c \subset \bar{E} \cap (\bar{E} \cap E)^c = \bar{E} \cap (\bar{E} \cup E) \). Hence
\[
\nu(E) \geq \mu(K) = \mu(\bar{E} \setminus O) \geq \mu(\bar{E} \setminus E) = \mu(E) + \mu(\bar{E} \setminus E) - \mu(O) \geq \mu(E) - \epsilon \quad \forall \epsilon > 0 \Rightarrow \nu(E) \geq \mu(E)
\]

(b) **Proof that** \( E = H \cup N \Rightarrow E \in M_\mu \): This is obvious because \( H \in B_{\mathbb{R}} \subset M_\mu \), \( N \in M_\mu \) by problem 1a and \( M_\mu \) is closed under finite unions.

**Proof that** \( E \in M_\mu \Rightarrow E = H \cup N \text{ when } \mu(E) < \infty \): By part (a), there is, for each \( j \in \mathbb{N} \) a compact set \( K_j \subset E \) with \( \mu(E) \leq \mu(K_j) + \frac{1}{2^j} \). Then \( H = \bigcup_{j=1}^{\infty} K_j \), \( N = E \setminus H \) does the job since, for every \( j \in \mathbb{N} \)
\[
\mu(N) = \mu(E \setminus H) \leq \mu(E \setminus K_j) \leq \frac{1}{2^j}
\]

**Proof that** \( E \in M_\mu \Rightarrow E = H \cup N \text{ for general } E \in M_\mu \): Let, for each \( n \in \mathbb{Z} \), \( E_n = E \cap (n, n+1] \). By the case just handled, \( E_n = H_n \cup \tilde{N}_n \) with \( H_n \) a countable union of compact sets and \( \tilde{N}_n \) a null set. Then \( H = \bigcup_{n=-\infty}^{\infty} H_n \) and \( N = \bigcup_{n=-\infty}^{\infty} \tilde{N}_n \) do the job, since a countable union of countable sets is countable and since the collection of null sets is closed under countable unions.

5. (a) Let \( A \) be an algebra generating a \( \sigma \)-algebra \( M \), and let \( \mu \) be a finite measure on \( M \). Prove that for any \( E \in M \) and any \( \epsilon > 0 \) there exists \( A \in A \) such that \( \mu(E \Delta A) < \epsilon \). (Hint: show that the collection of all \( E \) which can be approximated in this way is a \( \sigma \)-algebra).

(b) Let \( m \) be the Lebesgue measure and \( L \) be the collection of Lebesgue measurable sets. Prove that if \( E \in L \) and \( m(E) < \infty \) then for any \( \epsilon > 0 \) there exists a finite union of open intervals \( U \) such that \( m(E \Delta U) < \epsilon \).

**Solution.**

(a) Let \( M_\alpha \) be the set of all \( E \in M \) for which there exists, for each \( \epsilon > 0 \), an \( A \in A \) such that \( \mu(E \Delta A) < \epsilon \).

- If \( E \in A \), we can always choose \( A = E \), so \( A \subset M_\alpha \).
- We now verify that \( M_\alpha \) is closed under taking complements. Let \( E \in M_\alpha \), \( \epsilon > 0 \) and \( A \in A \) such that \( \mu(E \Delta A) < \epsilon \). Then \( A^c \in A \) and
\[
E^c \Delta A^c = (E^c \cap A) \cup (A^c \cap E) = E \Delta A \Rightarrow \mu(E^c \Delta A^c) = \mu(E \Delta A) < \epsilon \Rightarrow E^c \in M_\alpha
\]

2
Since the sum on the left converges, there is an $N$ such that $\mu(E_j \Delta A_j) < \frac{\varepsilon}{2^{j+1}}$. Let $E = \bigcup_{j=1}^{\infty} E_j$ and $\hat{A} = \bigcup_{j=1}^{\infty} A_j$.

Then

$$E \Delta \hat{A} = (E \cap \hat{A}^c) \cup (\hat{A} \cap E^c)$$

$$= \left[ \left( \bigcup_j E_j \right) \cap \left( \bigcap_j A_j^c \right) \right] \cup \left[ \left( \bigcup_j A_j \right) \cap \left( \bigcap_j E_j^c \right) \right]$$

$$\subseteq \left[ \bigcup_j (E_j \cap A_j^c) \right] \cup \left[ \bigcup_j (A_j \cap E_j^c) \right]$$

$$= \bigcup_{j=1}^{\infty} \left[ (E_j \cap A_j^c) \cup (A_j \cap E_j^c) \right]$$

$$= \bigcup_{j=1}^{\infty} (E_j \Delta A_j)$$

Hence

$$\mu(E \Delta \hat{A}) \leq \sum_{j=1}^{\infty} \mu(E_j \Delta A_j) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2}$$

As $\hat{A} \in \mathcal{M}$,

$$\mu(\hat{A}) = \lim_{n \to \infty} \mu\left( \bigcup_{j=1}^{n} A_j \right)$$

by continuity from below. Since $\mu(\hat{A}) < \infty$, there is an $n \in \mathbb{N}$ such that $0 < \mu(\hat{A}) - \mu\left( \bigcup_{j=1}^{n} A_j \right) < \frac{\varepsilon}{2}$. Set $A = \bigcup_{j=1}^{n} A_j$. Then $A \in \mathcal{A}$ and $A^c = \hat{A}^c \cup (\hat{A} \cap A^c)$ so that

$$E \Delta A = (E \cap A^c) \cup (A \cap E^c) \subset (E \cap \hat{A}^c) \cup (\hat{A} \cap A^c) \cup (\hat{A} \cap A^c) = (E \Delta \hat{A}) \cup (\hat{A} \cap A^c)$$

and

$$\mu(E \Delta A) \leq \mu(E \Delta \hat{A}) + \mu(\hat{A} \cap A^c) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Rightarrow E = \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}_a$$

Thus $\mathcal{M}_a$ is a $\sigma$-algebra that contains $\mathcal{A}$ and hence also the $\sigma$-algebra generated by $\mathcal{A}$, which is $\mathcal{M}$.

(b) We proved in class that

$$m(E) = \inf \left\{ \sum_{j=1}^{\infty} (b_j - a_j) \mid E \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}$$

So there is a countable collection of open intervals $\{(a_j, b_j)\}_{j \in \mathbb{N}}$ obeying $\hat{U} = \bigcup_{j=1}^{\infty} (a_j, b_j) \supset E$ and

$$\sum_{j=1}^{\infty} (b_j - a_j) \leq m(E) + \frac{\varepsilon}{2} \Rightarrow m(E) + m(\hat{U} \setminus E) = m(\hat{U}) \leq m(E) + \frac{\varepsilon}{2} \Rightarrow m(\hat{U} \setminus E) \leq \frac{\varepsilon}{2}$$

Since the sum on the left converges, there is an $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty} (b_j - a_j) < \frac{\varepsilon}{2}$. Then $U = \bigcup_{j=1}^{N} (a_j, b_j)$ does the job because

$$E \Delta U = (U \setminus E) \cup (E \setminus U) \subset (\hat{U} \setminus E) \cup (\hat{U} \setminus U) \Rightarrow m(E \Delta U) \leq m(\hat{U} \setminus E) + m(\hat{U} \setminus U) < \varepsilon$$