6 Signed Measures

Definition 6.1 (Signed Measures). Let $X$ be a nonempty set and $\mathcal{M} \subset \mathcal{P}(X)$ be a $\sigma$-algebra.

(a) A signed measure on $(X, \mathcal{M})$ is a function $\nu: \mathcal{M} \to [-\infty, \infty]$ such that

(i) $\nu(\emptyset) = 0$

(ii) $\nu$ assumes at most one\(^1\) of the values $\pm \infty$.

(iii) If $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ is disjoint, then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$

with the sum converging absolutely\(^2\) if $\nu\left(\bigcup_{n=1}^{\infty} E_n\right)$ is finite.

(b) A set $E \in \mathcal{M}$ is said to be positive (negative, null) for the signed measure $\nu$ if

$$F \in \mathcal{M}, \ F \subset E \implies \nu(F) \geq 0 \ (\leq 0, = 0)$$

Example 6.2.

(a) If $\mu$ is a measure on $(X, \mathcal{M})$, then $\mu$ is a signed measure and $X$ is a positive set for $\mu$.

(b) If $\mu$ and $\nu$ are measures on $(X, \mathcal{M})$, with at least one of them finite, then $\mu - \nu$ is a signed measure.

(c) If $\mu$ is a measure on $(X, \mathcal{M})$ and $f: X \to \mathbb{R}$ is measurable with at least one of $\int \max\{f, 0\} \, d\mu$ and $\int \max\{-f, 0\} \, d\mu$ finite, then

$$\nu(E) = \int_E f(x) \, d\mu(x) = \int_E \max\{f, 0\} \, d\mu - \int_E \max\{-f, 0\} \, d\mu$$

is a signed measure. We call $f$ an extended $\mu$-integrable function and we write $d\nu(x) = f(x) \, d\mu(x)$. A set $E \in \mathcal{M}$ is positive (negative, null) for $\nu$ if and only if $f(x) \geq 0 \ (\leq 0, = 0)$ almost everywhere on $E$.

Lemma 6.3 (Continuity). Let $\nu$ be a signed measure on $(X, \mathcal{M})$.

(a) If $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ with $E_1 \subset E_2 \subset E_3 \subset \cdots$, then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \nu(E_n)$$

\[\text{so that we don't have to deal with } -\infty - \infty\]

\[\text{so that we don't have to worry about the order of the terms in } \sum_{n=1}^{\infty} \nu(E_n)\]
(b) If \( \{ E_n \}_{n \in \mathbb{N}} \subset \mathcal{M} \) with \( E_1 \supset E_2 \supset E_3 \supset \cdots \) and \( \nu(E_1) \) is finite, then

\[
\nu \left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} \nu(E_n)
\]

**Proof.** Problem Set 10, #1.

**Lemma 6.4** (Positive sets). Let \( \nu \) be a signed measure on \( (X, \mathcal{M}) \).

(a) If \( E \) is a positive set for \( \nu \) and \( F \in \mathcal{M} \) with \( F \subset E \), then \( F \) is a positive set for \( \nu \).

(b) If, for each \( n \in \mathbb{N} \), \( E_n \) is a positive set for \( \nu \), then \( \bigcup_{n=1}^{\infty} E_n \) is a positive set for \( \nu \).

**Proof.** Problem Set 10, #2.

**Theorem 6.5** (The Hahn Decomposition Theorem). If \( \nu \) is a signed measure on \( (X, \mathcal{M}) \), then there is a positive set \( P \in \mathcal{M} \) and a negative set \( N \in \mathcal{M} \) for \( \nu \) such that \( P \cup N = X \) and \( P \cap N = \emptyset \). If \( P', N' \) is any other such pair of sets, then \( P \Delta P' = N \Delta N' \) is null.

**Proof.** Assume, without loss of generality, that \( \nu \) never takes the value \( +\infty \). (Otherwise consider \( -\nu \).)

**Step 1: Choice of \( P \):**

Set

\[
m = \sup \left\{ \nu(E) \mid E \text{ positive} \right\}
\]

Then there exists a sequence \( \{ \tilde{P}_n \}_{n \in \mathbb{N}} \) of positive sets such that

\[
\lim_{n \to \infty} \nu(\tilde{P}_n) = m
\]

Let \( P_n = \bigcup_{m=1}^{n} \tilde{P}_n \). Then \( P_1 \subset P_2 \subset P_3 \subset \cdots \), and each \( P_n \) is positive, and since

\[
\nu(P_n) = \nu(\tilde{P}_n) + \nu(P_n \setminus \tilde{P}_n),
\]

\[
(\nu(\tilde{P}_n) \leq \nu(P_n) \leq m)
\]

Set \( P = \bigcup_{n=1}^{\infty} P_n \). Then \( P \) is positive, and by continuity from below,

\[
\nu(P) = \lim_{n \to \infty} \nu(P_n) = m \implies m < \infty
\]

since, by hypothesis, \( \nu \) never takes the value \( +\infty \).
Step 2: Proof that $N \equiv X \setminus P$ is negative:

$N$ cannot contain a positive, nonnull subset $A$, because, if it did, $P \cup A$ would be a positive subset with $\nu(P \cup A) > \nu(P)$, contradicting the maximality of $\nu(P)$.

On the other hand, if $N$ fails to be negative, there must exist a subset $B \subseteq N$ with $B \in \mathcal{M}$ and $\nu(B) > 0$. Now it suffices to apply the lemma (proven below) that

$$B \in \mathcal{M}, \ 0 < \nu(B) < \infty \implies \exists A \in \mathcal{M} \text{ with } A \subset B \text{ and } A \text{ positive, nonnull}$$

Step 3: Uniqueness up to null sets:

If $P'$ is positive, $N'$ is negative and $P' \cap N' = \emptyset$, $P' \cup N' = X$, then

$P \setminus P'$ is positive (since $P \setminus P' \subset P$, positive)

is negative (since $P \setminus P' \subset X \setminus P' = N'$, negative)

$\implies P \setminus P'$ is null

Similarly $P' \setminus P$ is null. So

$$P \Delta P' = (P \cap P'^c) \cup (P^c \cap P') = (N^c \cap N') \cup (N \cap N'^c) = N \Delta N'$$

are null.

**Lemma 6.6.** Let $E \in \mathcal{M}$ with $0 < \nu(E) < \infty$. Then there exists an $A \in \mathcal{M}$ with $A \subset E$ that is positive and nonnull.

**Proof.**

- If $E$ is positive, choose $A = E$.
- Otherwise choose $E_1 \subset E$ with $E_1 \in \mathcal{M}$ and $\nu(E_1) < -\frac{1}{n_1}$ where $n_1$ is the smallest natural number for which such an $E_1$ exists.

  - $\nu(E \setminus E_1) > 0$, since $\nu(E_1) + \nu(E \setminus E_1) = \nu(E) > 0$.
  - If $E \setminus E_1$ is positive, choose $A = E \setminus E_1$, and we’re done.

- Otherwise, inductively choose $E_k \subset E \setminus \bigcup_{j=1}^{k-1} E_j$ with $E_k \in \mathcal{M}$ and with $\nu(E_k) < -\frac{1}{n_k}$ where $n_k$ is the smallest natural number for which such an $E_k$ exists.

  - $\nu(E \setminus \bigcup_{j=1}^{k} E_j) > 0$, since $\nu(E \setminus \bigcup_{j=1}^{k} E_j) + \sum_{j=1}^{k} \nu(E_j) = \nu(E) > 0$.
  - If $E \setminus \bigcup_{j=1}^{k} E_j$ is positive, choose $A = E \setminus \bigcup_{j=1}^{k} E_j$, and we’re done.
Otherwise, choose $A = E \setminus \bigcup_{j=1}^{\infty} E_j$.

Proof that $A$ is nonnull:

Since all of the subsets in the union $E = A \cup \left( \bigcup_{j=1}^{\infty} E_j \right)$ are disjoint

$$0 < \nu(E) = \nu(A) + \sum_{j=1}^{\infty} \nu(E_j)$$

so that $\nu(A) > 0$ and furthermore $\sum_{j=1}^{\infty} \nu(E_j)$ converges absolutely, which forces $\lim_{j \to \infty} n_j = \infty$.

Proof that $A$ is positive:

It suffices to prove that, for each $\varepsilon > 0$, there is no $B \subset A$ with $B \in \mathcal{M}$ and $\nu(B) < -\varepsilon$. Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ with $\frac{1}{n_k - 1} < \varepsilon$. Then

$$A = E \setminus \bigcup_{j=1}^{\infty} E_j \subset E \setminus \bigcup_{j=1}^{k-1} E_j$$

By construction, $n_k$ is the smallest natural number for which there is a $B \in \mathcal{M}$ with $B \subset E \setminus \bigcup_{j=1}^{k-1} E_j$ and $\nu(B) < -\frac{1}{n_k - 1}$. So there is no $B \in \mathcal{M}$ with $B \subset A$ and $\nu(B) < -\frac{1}{n_k - 1}$. So there is no $B \in \mathcal{M}$ with $B \subset A$ and $\nu(B) < -\varepsilon$.

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**Definition 6.7.** Two signed measure $\mu, \nu$ on $(X, \mathcal{M})$ are said to be **mutually singular**, denoted $\mu \perp \nu$, if there exist sets $E, F \in \mathcal{M}$ such that $E \cup F = X$, $E \cap F = \emptyset$, $E$ is null for $\mu$ and $F$ is null for $\nu$.

**Example 6.8** (mutually singular measures). Let $X = \mathbb{R}$ and $\mathcal{M} = \mathcal{L}$, the $\sigma$-algebra of Lebesgue measurable sets. Then the measures

$$\mu = m = \text{Lebesgue measure}$$

$$\nu = \text{point mass or Dirac measure at } 0$$

That is (see Example 2.15),

$$\nu(E) = \begin{cases} 
1 & \text{if } 0 \in E \\
0 & \text{if } 0 \notin E
\end{cases}$$

are mutually singular, because

$$\mu(\{0\}) = 0 \quad \text{and} \quad \nu(\mathbb{R} \setminus \{0\}) = 0$$
Theorem 6.9 (The Jordan Decomposition Theorem). If $\nu$ is a signed measure on $(X, \mathcal{M})$, then there are unique positive measures $\nu_+, \nu_-$ on $(X, \mathcal{M})$ such that $\nu_+ \perp \nu_-$ and $\nu = \nu_+ - \nu_-$. 

Proof. 

Existence: 
Let $X = P \cup N$ be the Hahn decomposition of Theorem 6.5. Then 

$$\nu_+(A) = \nu(A \cap P) \quad \nu_-(A) = -\nu(A \cap N)$$ 

works.

Uniqueness: 
Let $\nu = \mu_+ - \mu_-$ also work. Let $P', N'$ be such that $P' \cup N' = X$, $P' \cap N' = \emptyset$, $N'$ is null for $\mu_+$ and $P'$ is null for $\mu_-$. Then

$P'$ is positive for $\nu$

$N'$ is negative for $\nu$

$\implies P' \cup N'$ is another Hahn decomposition for $\nu$

$\implies P \Delta P'$ is null for $\nu$

So, if $A \in \mathcal{M}$,

$$\begin{align*}
\mu_+(A) &= \mu_+(A \cap P') + \mu_+(A \cap N') \\
&= \mu_+(A \cap P') - \mu_-(A \cap P') \\
&= \nu(A \cap P') + \nu(A \cap (P' \setminus P)) \\
&= \nu(A \cap P) \\
&= \nu_+(A \cap P) - \nu_-(A \cap P) = \nu_+(A \cap P)
\end{align*}$$

and, similarly, $\mu_-(A) = \nu_-(A)$. 

$\square$