Review of Signed Measures and the Radon–Nikodym Theorem

Let X be a nonempty set and $\mathcal{M} \subset \mathcal{P}(X)$ a σ–algebra.

Definition 1 (Signed Measures)

(a) A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

(i) $\nu(\emptyset) = 0$

(ii) ν assumes at most one of the values $\pm\infty$.

(iii) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ is disjoint, then

$$\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \nu(E_n)$$

with the sum converging absolutely if $\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big)$ is finite.

(b) A set $E \in \mathcal{M}$ is said to be positive (negative, null) for the signed measure ν if

$$F \in \mathcal{M},\ F \subset E \implies \nu(F) \ge 0\ (\le 0, = 0)$$

Example 2

(a) If μ is a measure on $(X, \mathcal{M})$, then μ is a signed measure and X is a positive set for μ.

(b) If μ and ν are measures on $(X, \mathcal{M})$, with at least one of them finite, then $\mu - \nu$ is a signed measure.

(c) If μ is a measure on $(X, \mathcal{M})$ and $f : X \to \mathbb{R}$ is measurable with at least one of $\int \max\{f, 0\} d\mu$ and $\int \max\{-f, 0\} d\mu$ finite, then $\nu(E) = \int_E f(x) d\mu(x) = \int_E \max\{f, 0\} d\mu - \int_E \max\{-f, 0\} d\mu$ is a signed measure. We call f an extended μ–integrable function and we write $d\nu(x) = f(x) d\mu(x)$. A set $E \in \mathcal{M}$ is positive (negative, null) for ν if and only if $f(x) \ge 0$ (≤ 0, $= 0$) almost everywhere on E.

Lemma 3 (Continuity) *Let ν be a signed measure on $(X, \mathcal{M})$.*

(a) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ with $E_1 \subset E_2 \subset E_3 \subset \cdots$, then

$$\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \nu(E_n)$$

(b) If $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ with $E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\nu(E_1)$ is finite, then

$$\nu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \nu(E_n)$$

Lemma 4 (Positive sets) *Let ν be a signed measure on $(X, \mathcal{M})$.*

(a) If E is a positive set for ν and $F \in \mathcal{M}$ with $F \subset E$, then F is a positive set for ν.

(b) If, for each $n \in \mathbb{N}$, E_n is a positive set for ν, then $\bigcup_{n=1}^{\infty} E_n$ is a positive set for ν.

Theorem 5 (The Hahn Decomposition Theorem) *If ν is a signed measure on $(X, \mathcal{M})$, then there is a positive set $P \in \mathcal{M}$ and a negative set $N \in \mathcal{M}$ for ν such that $P \cup N = X$ and $P \cap N = \emptyset$. If P', N' is any other such pair of sets, then $P \Delta P' = N \Delta N'$ is null.*

Definition 6

(a) Two signed measure μ, ν on $(X, \mathcal{M})$ are said to be **mutually singular**, denoted $\mu \perp \nu$, if there exist sets $E, F \in \mathcal{M}$ such that $E \cup F = X$, $E \cap F = \emptyset$, E is null for μ and F is null for ν.

(b) Let ν be a signed measure on $(X, \mathcal{M})$ and μ be a positive measure on $(X, \mathcal{M})$. Then ν is said to be **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if

$$E \in \mathcal{M}, \mu(E) = 0 \implies \nu(E) = 0$$

Theorem 7 (The Jordan Decomposition Theorem) *If ν is a signed measure on $(X, \mathcal{M})$, then there are unique positive measures ν_+, ν_- on $(X, \mathcal{M})$ such that $\nu_+ \perp \nu_-$ and $\nu = \nu_+ - \nu_-$.*

Theorem 8 *Let ν be a finite signed measure (i.e. ν_+ and ν_- are both finite) and μ a positive measure on $(X, \mathcal{M})$. Then*

$$\nu \ll \mu \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu|(E) < \varepsilon$$

Here $|\nu|$ is the measure $\nu_+ + \nu_-$.

Corollary 9 *Let $f \in L^1(\mu)$. Then*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } E \in \mathcal{M}, \ \mu(E) < \delta \implies \left| \int_E f(x) \ d\mu(x) \right| < \varepsilon$$

Theorem 10 (The Radon–Nikodym Theorem) *Let ν be a σ–finite signed measure and μ be a σ–finite positive measure on $(X, \mathcal{M})$. There exist unique σ–finite signed measures λ, ρ on $(X, \mathcal{M})$ such that*

$$\lambda \perp \mu \qquad \rho \ll \mu \qquad \nu = \lambda + \rho$$

Moreover, there is an extended μ–integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$. Any two such functions are equal almost everywhere.