7 The Radon-Nikodym Theorem

Let \((X, \mathcal{M}, \mu)\) be a measure space. One easy way to build another measure from \(\mu\) is to take a measurable nonnegative function \(f\) and set

\[\nu(E) = \int_E f(x) \, d\mu(x) \quad \text{for all } E \in \mathcal{M}\]

In this case, one usually writes \(d\nu = f \, d\mu\).

Now suppose that someone gives you two measures, \(\mu\) and \(\nu\), on \(\mathcal{M}\). How can you tell if \(d\nu = f \, d\mu\)? There is one obvious necessary condition. If \(d\nu = f \, d\mu\), for some function \(f\) (even if you don’t know what it is), and if \(E \in \mathcal{M}\) is a null set for \(\mu\) (i.e. \(\mu(E) = 0\)), then

\[\nu(E) = \int_E f(x) \, d\mu(x) = \int_{\chi_E} f \, d\mu = \int 0 \, d\mu = 0\]

This property is given a name.

**Definition 7.1.** Let \(\nu\) be a signed measure on \((X, \mathcal{M})\) and \(\mu\) be a (positive) measure on \((X, \mathcal{M})\). Then \(\nu\) is said to be absolutely continuous with respect to \(\mu\), denoted \(\nu \ll \mu\), if

\[E \in \mathcal{M}, \ \mu(E) = 0 \implies \nu(E) = 0\]

In this part of the course we prove the Radon-Nikodym theorem, which says that, for \(\sigma\)-finite measures, absolute continuity is also a sufficient condition for \(\nu\) to be of the form \(d\nu = f \, d\mu\). Here is some prep.

**Proposition 7.2.** Let \(\nu\) be a finite signed measure (i.e. \(\nu_+\) and \(\nu_-\) are both finite) and \(\mu\) a (positive) measure on \((X, \mathcal{M})\). Then

\[\nu \ll \mu \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E)| < \varepsilon\]

**Proof.**

\emph{Part 1: Reduction to the case that \(\nu\) is positive:}

Use \(|\nu|\) to denote the measure \(\nu_+ + \nu_-\). Let \(X = P \cup N\) be a Hahn decomposition (of Theorem 6.5) for \(\nu\). So

\[\nu_+(E) = \nu(E \cap P) \quad \nu_-(E) = -\nu(E \cap N)\]
For the left hand side of the statement of the theorem

\[ \nu \ll \mu \iff \begin{cases} \mu(E) = 0 & \Rightarrow \nu(E) = 0 \\ \mu(E) = 0 & \Rightarrow \nu_+(E) = \nu_-(E) = 0 \\ \mu(E) = 0 & \Rightarrow |\nu|(E) = 0 \\ |\nu| \ll \mu \end{cases} \]

Call the statement on the right hand side of the theorem \( S_\nu \). That is, \( S_\nu \) is "\( \forall \varepsilon > 0 \ \exists \delta > 0 \) such that \( E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E)| < \varepsilon \)". We now show that \( S_\nu \) is true if and only if \( S_{|\nu|} \) is true.

\[
S_\nu \implies \left( \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E)| < \varepsilon \right)
\]

(just replaced \( \varepsilon \) by \( \frac{\varepsilon}{2} \))

\[
\implies \left( \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E \cap P)|, |\nu(E \cap N)| < \frac{\varepsilon}{2} \right)
\]

(since \( \mu(E) < \delta \) implies \( \mu(E \cap N), \mu(E \cap P) < \delta \))

\[
\implies \left( \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu|(E) < \varepsilon \right)
\]

\[
\implies S_{|\nu|}
\]

and

\[
S_{|\nu|} \implies \left( \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu|(E) < \varepsilon \right)
\]

\[
\implies \left( \forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu|(E)| < \varepsilon \right)
\]

(since \( |\nu(E)| \leq |\nu|(E) \))

\[
\implies S_\nu
\]

**Part 2: Proof of \( \leftarrow \) in the case that \( \nu \) is positive:**

We are assuming that \( S_\nu \) is true and have to prove that \( \nu \ll \mu \). Let \( E \in \mathcal{M} \). Then

\[
\mu(E) = 0 \implies |\nu(E)| < \varepsilon \text{ for all } \varepsilon > 0
\]

\[
\implies \nu(E) = 0
\]

so that \( \nu \ll \mu \).

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Part 3: Proof of \( \implies \) in the case that \( \nu \) is positive:

We are assuming that \( \nu \ll \mu \) and have to prove that \( S_\nu \) is true. Assume that \( S_\nu \) is false. Then there is an \( \varepsilon > 0 \) and, for every \( n \in \mathbb{N} \), there is an \( E_n \in \mathcal{M} \) such that \( \mu(E_n) < \frac{1}{2^n} \) and \( \nu(E_n) \geq \varepsilon \). Set

\[
F_k = \bigcup_{n=k}^{\infty} E_n \quad F = \bigcap_{k=1}^{\infty} F_k
\]

Then

\[
\mu(F_k) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{2}{2^k} \implies \mu(F) = \lim_{k \to \infty} \mu(F_k) = 0
\]

but

\[
\nu(F_k) \geq \nu(E_k) \geq \varepsilon \stackrel{\text{finite!}}{\implies} \nu(F) = \lim_{k \to \infty} \nu(F_k) \geq \varepsilon
\]

contradicting the assumption that \( \nu \ll \mu \).

\[ \square \]

**Corollary 7.3.** Let \( f \in L^1(\mu) \). Then

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } E \in \mathcal{M}, \ \mu(E) < \delta \implies \left| \int_E f(x) \ d\mu(x) \right| < \varepsilon
\]

**Proof.** Apply Proposition 7.2 with \( \nu(E) = \int_E f(x) \ d\mu(x) \).

\[ \square \]

**THEOREM 7.4** (The Radon-Nikodym Theorem). Let \( \nu \) be a \( \sigma \)-finite signed measure and \( \mu \) be a \( \sigma \)-finite (positive) measure on \( (X, \mathcal{M}) \). There exist unique \( \sigma \)-finite signed measures \( \lambda, \rho \) on \( (X, \mathcal{M}) \) such that

\[
\lambda \perp \mu, \quad \rho \ll \mu, \quad \nu = \lambda + \rho
\]

Moreover, there is an extended \( \mu \)-integrable function \( f : X \to \mathbb{R} \) such that \( d\rho = f \ d\mu \).

Any two such functions are equal almost everywhere.

First, we have a preliminary

**Lemma 7.5.** Let \( \mu, \nu \) be finite (positive) measures on \( (X, \mathcal{M}) \). Then either \( \nu \perp \mu \) or

\[
\exists \varepsilon > 0, \ E \in \mathcal{M} \text{ such that } \mu(E) > 0 \text{ and } \nu \geq \varepsilon \mu \text{ on } E
\]

That is, \( \mu(E) > 0 \) and \( E \) is a positive set for \( \nu - \varepsilon \mu \).

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Proof. Let, for each \( n \in \mathbb{N} \), \( X = P_n \cup N_n \) be a Hahn decomposition for \( \nu - \frac{1}{n} \mu \). Let

\[
P = \bigcup_{n=1}^{\infty} P_n \quad N = \bigcap_{n=1}^{\infty} N_n = P^c
\]

Then

For every \( n \in \mathbb{N} \), \( N_n \) is a negative set for \( \nu - \frac{1}{n} \mu \)

\[
\Rightarrow N \text{ is a negative set for } \nu - \frac{1}{n} \mu \text{ for all } n \in \mathbb{N} \\
\Rightarrow \nu(N) - \frac{1}{n} \mu(N) \leq 0 \text{ for all } n \in \mathbb{N} \\
\Rightarrow 0 \leq \nu(N) \leq \frac{1}{n} \mu(N) \text{ for all } n \in \mathbb{N} \\
\Rightarrow \nu(N) = 0
\]

If \( \mu(P) = 0 \), then \( \nu \perp \mu \).
If \( \mu(P) > 0 \), then, by continuity from below, \( \mu(P_n) > 0 \) for some \( n \in \mathbb{N} \), and then \( P_n \) is a positive set for \( \nu - \frac{1}{n} \mu \).

\[\square\]

Proof of Theorem 7.4.
Part 1: A little motivation in the case that \( \mu, \nu \) are finite and positive:
We want to guess a function \( f \) so that, if we set \( d\rho = f \, d\mu \), then \( \lambda = \nu - \rho \) is mutually singular with respect to \( \mu \). So let’s suppose that we already know that

\[
\nu = \lambda + \rho \\
d\rho = f \, d\mu \\
\lambda \perp \mu \\
X = L \cup M \\
L \cap M = \emptyset \\
\lambda(M) = 0 \\
\mu(L) = 0
\]

for some (unknown) \( \lambda, \rho, L, M \), and see what conditions (that do not mention \( \lambda, \rho, L, M \) — which we do not yet know) \( f \) has to obey.

- Since \( \nu \) is positive and \( \lambda(E) = \lambda(E \cap L) = \nu(E \cap L) \geq 0 \),

\[
\nu(E) = \lambda(E) + \int_E f \, d\mu \geq \int_E f \, d\mu \text{ for all } E \in \mathcal{M}
\]

and we’ll use

\[
\mathcal{F} = \left\{ \varphi : X \to [0, \infty], \text{ measurable} \middle| \int_E \varphi \, d\mu \leq \nu(E) \ \forall E \in \mathcal{M} \right\}
\]

as the set of possible candidates for the function \( f \).
Think of $\nu(X)$, $\lambda(X) = \nu(L)$ and $\rho(X) = \nu(M)$ as being the “mass” of $X$, $L$ and $M$, respectively. We will try to pick $f$ to be the candidate $\varphi \in \mathcal{F}$ which puts as much mass, $\int_X \varphi \, d\mu$, as possible into $\rho(X)$. The remaining mass, which cannot be put into $\rho(X)$ because it is not in $M$, will go into $\lambda(X)$, i.e. into $L$.

**Part 2: Proof of existence in the case that $\mu$, $\nu$ are finite and positive:**

The set $\mathcal{F}$ of candidates for the function $f$ has the properties

- $\mathcal{F} \neq \emptyset$ since $\varphi = 0 \in \mathcal{F}$
- if $\varphi, \psi \in \mathcal{F}$, then $\zeta = \max\{\varphi, \psi\} \in \mathcal{F}$ since, writing $A = \{x \in X \mid \varphi(x) > \psi(x)\}$,

$$\int_E \zeta \, d\mu = \int_{A \cap E} \varphi \, d\mu + \int_{E \setminus A} \psi \, d\mu \leq \nu(A \cap E) + \nu(E \setminus A) = \nu(E)$$

Set

$$a = \sup \left\{ \int_X \varphi \, d\mu \mid \varphi \in \mathcal{F} \right\} \leq \nu(X) < \infty$$

and choose $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ such that $\lim_{n \to \infty} \int_X \varphi_n \, d\mu = a$. Then set

$$g_n(x) = \max\{\varphi_1(x), \ldots, \varphi_n(x)\} \quad f(x) = \sup_n \varphi_n(x)$$

Then

- $g_n \in \mathcal{F}$, for all $n \in \mathbb{N}$
- $g_n(x)$ increases to $f(x)$, as $n \to \infty$, for all $x \in X$ and
- $a \geq \int_X g_n \, d\mu \geq \int_X \varphi_n \, d\mu \xrightarrow{n \to \infty} a$

which forces

$$\lim_{n \to \infty} \int_X g_n \, d\mu = a$$

By the monotone convergence theorem

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E g_n \, d\mu \leq \nu(E)$$

So

$$f \in \mathcal{F} \quad \int_X f \, d\mu = a < \infty \quad 0 \leq f < \infty \text{ a.e.}$$

Now we set

$$d\rho = f \, d\mu \quad \lambda = \nu - \rho$$

Then

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\( \nu = \lambda + \rho \)
\( \rho \ll \mu, \rho \) is positive
\( \lambda \) is positive, since \( f \in F \) so that \( \rho(E) = \int_E f \, d\mu \leq \nu(E) \) for all \( E \in M \)
\( \lambda \perp \mu \)

**Proof:** If not, then by the preliminary Lemma 7.5, there is an \( \varepsilon > 0 \) and an \( E_0 \in M \) with \( \mu(E_0) > 0 \) and \( \lambda \geq \varepsilon \mu \) on \( E_0 \). Set \( d\rho' = \varepsilon \chi_{E_0} \, d\mu \). Then

\[
\rho' \leq \lambda = \nu - \rho, \text{ or } \rho + \rho' \leq \nu.
\]

That is \( (f + \varepsilon \chi_{E_0}) \, d\mu \leq d\nu \) or \( f + \varepsilon \chi_{E_0} \in F \) and

\[
\int_X (f + \varepsilon \chi_{E_0}) \, d\mu = a + \varepsilon \mu(E_0) > a
\]

which contradicts the definition of \( a \).

**Part 3: Proof of uniqueness in the case that \( \mu, \nu \) are finite and positive:**

Suppose that \( \nu = \lambda + \rho \), \( \rho = \rho' \) and \( f = f' \) a.e.. We have \( \lambda = \lambda' = \rho' - \rho \) and

\( \lambda - \lambda' \perp \mu \) since \( \mu(E \cup E') = 0 \) and \( (\lambda - \lambda')(G) = 0 \) for all \( G \subseteq (E \cup E')^c \).

\( \rho' - \rho = (f' - f) \, d\mu \ll \mu \)

which implies that, for all \( S \in M \),

\[
(\lambda - \lambda')(S) = (\lambda - \lambda')(S \cap (E \cup E'))
\]

\[
= (\rho' - \rho)(S \cap (E \cup E'))
\]

\[
= 0 \quad \text{since } \mu(S \cap (E \cup E')) = 0
\]

\( \implies (\rho' - \rho)(S) = 0 \)

which implies that \( \lambda = \lambda' \) and that \( \rho = \rho' \). Finally, to show that \( f' - f = 0 \) a.e., set, for each \( n \in \mathbb{N} \), \( P_n = \{ x \mid f'(x) \geq f(x) + 1/n \} \) and \( N_n = \{ x \mid f(x) \geq f'(x) + 1/n \} \).
Then
\[(\rho' - \rho)(P_n) = 0 \implies \int_{P_n} (f' - f) \, d\mu = 0 \implies \mu(P_n) = 0\]
\[(\rho' - \rho)(N_n) = 0 \implies \int_{N_n} (f' - f) \, d\mu = 0 \implies \mu(N_n) = 0\]

**Part 4: Proof in the case that \(\mu, \nu\) are \(\sigma\)-finite and positive:**

Write
\[X = \bigcup_{m=1}^{\infty} E_m\quad\text{with the } E_m\text{'s disjoint and each } \mu(E_m) < \infty\]
\[X = \bigcup_{n=1}^{\infty} F_n\quad\text{with the } F_n\text{'s disjoint and each } \nu(F_n) < \infty\]

Then
\[X = \bigcup_{m,n=1}^{\infty} E_m \cap F_n\]

with the \(E_m \cap F_n\)'s all disjoint and each \(\mu(E_m \cap F_n), \nu(E_m \cap F_n) < \infty\). Set, for each \(m, n \in \mathbb{N}\),
\[\mu_{m,n}(A) = \mu(A \cap E_m \cap F_n)\]
\[\nu_{m,n}(A) = \nu(A \cap E_m \cap F_n)\]

By the previous step, we have
\[d\nu_{m,n} = d\lambda_{m,n} + f_{m,n} \, d\mu_{m,n} \quad\text{with } \lambda_{m,n} \perp \mu_{m,n}\]

It now suffices to set
\[\lambda = \sum_{m,n=1}^{\infty} \lambda_{m,n} \quad f = \sum_{m,n=1}^{\infty} f_{m,n} \, \chi_{E_m \cap F_n}\]
Part 4: Proof in the general case:

Just apply the previous case separately to $\nu_+$ and $\nu_-$. 

\[ \square \]

Remark 7.6.

Here is some notation and terminology.

(a) The decomposition $\nu = \lambda + \rho$ with $\lambda \perp \mu$ and $\rho \ll \mu$ is called the Lebesgue decomposition of $\nu$ with respect to $\mu$.

(b) If $\nu \ll \mu$, then $d\nu = f \, d\mu$ for some $f$. This is called the Radon-Nikodym theorem and we write $f = \frac{d\nu}{d\mu}$.