7 The Radon-Nikodym Theorem

Let \((X, \mathcal{M}, \mu)\) be a measure space. One easy way to build another measure from \(\mu\) is to take a measurable nonnegative function \(f\) and set

\[
\nu(E) = \int_E f(x) \, d\mu(x) \quad \text{for all } E \in \mathcal{M}
\]

In this case, one usually writes \(d\nu = f \, d\mu\).

Now suppose that someone gives you two measures, \(\mu\) and \(\nu\), on \(\mathcal{M}\). How can you tell if \(d\nu = f \, d\mu\) for some function \(f\) (even if you don’t know what it is), and if \(E \in \mathcal{M}\) is a null set for \(\mu\) (i.e. \(\mu(E) = 0\)), then

\[
\nu(E) = \int_E f(x) \, d\mu(x) = \int f \chi_E \, d\mu = \int 0 \, d\mu = 0
\]

This property is given a name.

**Definition 7.1.** Let \(\nu\) be a signed measure on \((X, \mathcal{M})\) and \(\mu\) be a positive measure on \((X, \mathcal{M})\). Then \(\nu\) is said to be absolutely continuous with respect to \(\mu\), denoted \(\nu \ll \mu\), if

\[
E \in \mathcal{M}, \ \mu(E) = 0 \implies \nu(E) = 0
\]

In this part of the course we prove the Radon-Nikodym theorem, which says that, for \(\sigma\)-finite measures, absolute continuity is also a sufficient condition for \(\nu\) to be of the form \(d\nu = f \, d\mu\). Here is some prep.

**Theorem 7.2.** Let \(\nu\) be a finite signed measure (i.e. \(\nu_+\) and \(\nu_-\) are both finite) and \(\mu\) a positive measure on \((X, \mathcal{M})\). Then

\[
\nu \ll \mu \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{such that} \ E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E)| < \varepsilon
\]

**Proof.**

**Part 1: Reduction to the case that \(\nu\) is positive:**

Use \(|\nu|\) to denote the measure \(\nu_+ + \nu_-\). Let \(X = P \cup N\) be a Hahn decomposition (of Theorem 6.5) for \(\nu\). So

\[
\nu_+(E) = \nu(E \cap P) \quad \nu_-(E) = -\nu(E \cap N)
\]
For the left hand side of the statement of the theorem
\[ \nu \preceq \mu \iff \begin{cases} \mu(E) = 0 \implies \nu(E) = 0 \end{cases} \]
\[ \iff \begin{cases} \mu(E) = 0 \implies \nu_+(E) = \nu_-(E) = 0 \end{cases} \]
\[ \iff \begin{cases} \mu(E) = 0 \implies |\nu|(E) = 0 \end{cases} \]
\[ \iff |\nu| \preceq \mu \]

Call the statement on the right hand side of the theorem $S_\nu$. That is, $S_\nu$ is "\( \forall \varepsilon > 0 \ \exists \delta > 0\) such that \( E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E)| < \varepsilon \)". We now show that $S_\nu$ is true if and only if $S_{|\nu|}$ is true.

\[ S_\nu \implies \left( \forall \varepsilon > 0 \ \exists \delta > 0\) such that \( E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E)| < \frac{\varepsilon}{2} \right) \]
\[ \text{(just replaced } \varepsilon \text{ by } \frac{\varepsilon}{2}) \]
\[ \implies \left( \forall \varepsilon > 0 \ \exists \delta > 0\) s.t. \( E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu(E \cap N)|, |\nu(E \cap P) < \frac{\varepsilon}{2} \right) \]
\[ \text{(since } \mu(E) < \delta \text{ implies } \mu(E \cap N), \mu(E \cap P) < \delta) \]
\[ \implies \left( \forall \varepsilon > 0 \ \exists \delta > 0\) s.t. \( E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu|(E) < \varepsilon \right) \]
\[ \implies S_{|\nu|} \]

and

\[ S_{|\nu|} \implies \left( \forall \varepsilon > 0 \ \exists \delta > 0\) such that \( E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu|(E) < \varepsilon \right) \]
\[ \implies \left( \forall \varepsilon > 0 \ \exists \delta > 0\) such that \( E \in \mathcal{M}, \ \mu(E) < \delta \implies |\nu|(E) < \varepsilon \right) \]
\[ \text{(since } |\nu(E)| \leq |\nu|(E)) \]
\[ \implies S_\nu \]

**Part 2: Proof of \( \iff \) in the case that \( \nu \) is positive:**

We are assuming that $S_\nu$ is true and have to prove that $\nu \preceq \mu$. Let $E \in \mathcal{M}$. Then

\[ \mu(E) = 0 \implies |\nu(E)| < \varepsilon \text{ for all } \varepsilon > 0 \]
\[ \implies \nu(E) = 0 \]

so that $\nu \preceq \mu$. 

Part 3: Proof of \( \implies \) in the case that \( \nu \) is positive:

We are assuming that \( \nu \ll \mu \) and have to prove that \( S_\nu \) is true. Assume that \( S_\nu \) is false. Then there is an \( \varepsilon > 0 \) and, for every \( n \in \mathbb{N} \), there is an \( E_n \in \mathcal{M} \) such that \( \mu(E_n) < \frac{1}{2^n} \) and \( \nu(E_n) \geq \varepsilon \). Set

\[
F_k = \bigcup_{n=k}^{\infty} E_n, \quad F = \bigcap_{k=1}^{\infty} F_k
\]

Then

\[
\mu(F_k) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{2}{2^k} \implies \mu(F) = \lim_{k \to \infty} \mu(F_k) = 0
\]

but

\[
\nu(F_k) \geq \nu(E_k) \geq \varepsilon \implies \nu(F) = \lim_{k \to \infty} \nu(F_k) \geq \varepsilon
\]

contradicting the assumption that \( \nu \ll \mu \).

\[\square\]

**Corollary 7.3.** Let \( f \in L^1(\mu) \). Then

\[\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } E \in \mathcal{M}, \ \mu(E) < \delta \implies \left| \int_E f(x) \, d\mu(x) \right| < \varepsilon\]

**Proof.** Apply Theorem 7.2 with \( \nu(E) = \int_E f(x) \, d\mu(x) \).

\[\square\]

**Theorem 7.4 (The Radon-Nikodym Theorem).** Let \( \nu \) be a \( \sigma \)-finite signed measure and \( \mu \) be a \( \sigma \)-finite positive measure on \( (X, \mathcal{M}) \). There exist unique \( \sigma \)-finite signed measures \( \lambda, \rho \) on \( (X, \mathcal{M}) \) such that

\[
\lambda \perp \mu \quad \rho \ll \mu \quad \nu = \lambda + \rho
\]

Moreover, there is an extended \( \mu \)-integrable function \( f : X \to \mathbb{R} \) such that \( d\rho = f \, d\mu \). Any two such functions are equal almost everywhere.

First, we have a preliminary

**Lemma 7.5.** Let \( \mu, \nu \) be finite (positive) measures on \( (X, \mathcal{M}) \). Then either \( \nu \perp \mu \) or

\[\exists \varepsilon > 0, \ E \in \mathcal{M} \text{ such that } \mu(E) > 0 \text{ and } \nu \geq \varepsilon \mu \text{ on } E\]

That is, \( \mu(E) > 0 \) and \( E \) is a positive set for \( \nu - \varepsilon \mu \).
Proof. Let, for each $n \in \mathbb{N}$, $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$. Let

$$P = \bigcup_{n=1}^{\infty} P_n \quad N = \bigcap_{n=1}^{\infty} N_n = P^c$$

Then

- $N$ is a negative set for $\nu - \frac{1}{n}\mu$ for all $n \in \mathbb{N}$
- $\nu(N) - \frac{1}{n}\mu(N) \leq 0$ for all $n \in \mathbb{N}$
- $0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$ for all $n \in \mathbb{N}$
- $\nu(N) = 0$

If $\mu(P) = 0$, then $\nu \perp \mu$.
If $\mu(P) > 0$, then $\mu(P_n) > 0$ for some $n \in \mathbb{N}$, and then $P_n$ is a positive set for $\nu - \frac{1}{n}\mu$. \hfill \Box

Proof of Theorem 7.4.

Part 1: A little motivation in the case that $\mu, \nu$ are finite and positive:

We want to guess a function $f$ with $\lambda = \nu - f \, d\mu$ being mutually singular with respect to $\mu$. So let’s suppose that we already know that

$$\nu = \lambda + \rho \quad d\rho = f \, d\mu \quad \lambda \perp \mu$$

$$X = L \cup M \quad L \cap M = \emptyset \quad \lambda(M) = 0 \quad \mu(L) = 0$$

for some $\lambda, \rho, L, M$, and see what conditions (that do not mention $\lambda, \rho, L, M$ — which we do not yet know) $f$ has to obey.

- Since $\nu$ is positive and $\lambda \perp \rho$ (so that $\lambda$ is the restriction of $\nu$ to a subset of $X$), $\lambda$ is also positive. So

$$\nu(E) = \lambda(E) + \int_E f \, d\mu \geq \int_E f \, d\mu \quad \text{for all } E \in \mathcal{M}$$

and we’ll use

$$\mathcal{F} = \left\{ \varphi : X \to [0, \infty], \text{measurable} \mid \int_E \varphi \, d\mu \leq \nu(E) \ \forall E \in \mathcal{M} \right\}$$

as the set of possible candidates for the function $f$.

- Think of $\nu(X), \lambda(X) = \nu(L)$ and $\rho(X) = \nu(M)$ as being the “mass” of $X, L$ and $M$, respectively. We will try to pick $f$ to be the candidate $\varphi \in \mathcal{F}$ which puts as much mass, $\int_X \varphi \, d\mu$, as possible into $\rho(X)$. The remaining mass, which cannot be put into $\rho(X)$ because it is not in $M$, will go into $\lambda(X)$, i.e. into $L$. 

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Part 2: Proof of existence in the case that \( \mu, \nu \) are finite and positive:

The set \( \mathcal{F} \) of candidates for the function \( f \) has the properties

- \( \mathcal{F} \neq \emptyset \) since \( \varphi = 0 \in \mathcal{F} \)
- if \( \varphi, \psi \in \mathcal{F} \), then \( \zeta = \max\{\varphi, \psi\} \in \mathcal{F} \) since, writing \( A = \{x \in X | \varphi(x) > \psi(x)\} \),

\[
\int_E \zeta \, d\mu = \int_{A \cap E} \varphi \, d\mu + \int_{E \setminus A} \psi \, d\mu \\
\leq \nu(A \cap E) + \nu(E \setminus A) = \nu(E)
\]

Set

\[
a = \sup \left\{ \int_X \varphi \, d\mu \middle| \varphi \in \mathcal{F} \right\} \leq \nu(X) < \infty
\]

and choose \( \{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{F} \) such that \( \lim_{n \to \infty} \int_X \varphi_n \, d\mu = a \). Then set

\[
g_n(x) = \max\{\varphi_1(x), \ldots, \varphi_n(x)\} \quad f(x) = \sup_n \varphi_n(x)
\]

Then

- \( g_n \in \mathcal{F}, \) for all \( n \in \mathbb{N} \)
- \( g_n(x) \) increases to \( f(x) \), as \( n \to \infty \), for all \( x \in X \) and
- \( a \geq \int_X g_n \, d\mu \geq \int_X \varphi_n \, d\mu \xrightarrow{n \to \infty} a \)

which forces

\[
\lim_{n \to \infty} \int_X g_n \, d\mu = a
\]

By the monotone convergence theorem

\[
\int_E f \, d\mu = \lim_{n \to \infty} \int_E g_n \, d\mu \leq \nu(E)
\]

So

\[
f \in \mathcal{F} \quad \int_X f \, d\mu = a < \infty \quad 0 \leq f < \infty \text{ a.e.}
\]

Now we set

\[
d\rho = f \, d\mu \quad \lambda = \nu - \rho
\]

Then

- \( \nu = \lambda + \rho \)

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\[ \rho \ll \mu, \rho \text{ is positive} \]
\[ \lambda \text{ is positive, since } f \in \mathcal{F} \text{ so that } \rho(E) = \int_E f \, d\mu \leq \nu(E) \text{ for all } E \in \mathcal{M} \]
\[ \lambda \perp \mu \]

**Proof:** If not, then by the preliminary Lemma 7.5, there is an \( \varepsilon > 0 \) and an \( E_0 \in \mathcal{M} \) with \( \mu(E_0) > 0 \) and \( \lambda \geq \varepsilon \mu \) on \( E_0 \). Set \( d\rho' = \varepsilon \chi_{E_0} \, d\mu \). Then
\[ \rho' \leq \lambda = \nu - \rho, \text{ or } \rho + \rho' \leq \nu. \]
That is
\[ (f + \varepsilon \chi_{E_0}) \, d\mu \leq \nu \] for all \( E \in \mathcal{M} \)
which contradicts the definition of \( \alpha \).

**Part 3: Proof of uniqueness in the case that \( \mu, \nu \) are finite and positive:**
Suppose that
\[ \nu = \lambda + \rho \quad d\rho = f \, d\mu \quad X = E \cup F \quad E \cap F = \emptyset \quad \mu(E) = 0 \quad \lambda(F) = 0 \]
\[ \nu = \lambda' + \rho' \quad d\rho' = f' \, d\mu \quad X = E' \cup F' \quad E' \cap F' = \emptyset \quad \mu(E') = 0 \quad \lambda'(F') = 0 \]
We want to show that \( \alpha = \lambda', \rho = \rho' \) and \( f = f' \) a.e.. We have \( \lambda - \lambda' = \rho' - \rho \) and
\[ \lambda - \lambda' \perp \mu \text{ since } \mu(E \cup E') = 0 \text{ and } (\lambda - \lambda')(G) = 0 \text{ for all } G \subset (E \cup E')^c \]
\[ \rho' - \rho = (f' - f) \, d\mu \ll \mu \]
which implies that, for all \( S \in \mathcal{M} \),
\[ (\lambda - \lambda')(S) = (\lambda - \lambda')(S \cap (E \cup E')) \]
\[ = (\rho - \rho')(S \cap (E \cup E')) \]
\[ = 0 \text{ since } \mu(S \cap (E \cup E')) = 0 \]
which implies that \( \lambda = \lambda' \) and that \( \rho = \rho' \) and that \( f' - f - 0 \) a.e. since, if, for each \( n \in \mathbb{N} \), \( P_n = \{ x \mid f'(x) \geq f(x) + \frac{1}{n} \} \) and \( N_n = \{ x \mid f(x) \geq f'(x) + \frac{1}{n} \} \), then
\[ (\rho' - \rho)(P_n) = 0 \implies \int_{P_n}^{\geq 1/n} (f' - f) \, d\mu = 0 \implies \mu(P_n) = 0 \]
\[ (\rho' - \rho)(N_n) = 0 \implies \int_{N_n}^{\leq -1/n} (f' - f) \, d\mu = 0 \implies \mu(N_n) = 0 \]
Part 4: Proof in the case that $\mu, \nu$ are $\sigma$-finite and positive:

Write

\[ X = \bigcup_{m=1}^{\infty} E_m \] with the $E_m$'s disjoint and each $\mu(E_m) < \infty$

\[ X = \bigcup_{n=1}^{\infty} F_n \] with the $F_n$'s disjoint and each $\nu(F_n) < \infty$

Then

\[ X = \bigcup_{m,n=1}^{\infty} E_m \cap F_n \]

with the $E_m \cap F_n$'s all disjoint and each $\mu(E_m \cap F_n), \nu(E_m \cap F_n) < \infty$. Set, for each $m, n \in \mathbb{N}$,

\[
\begin{align*}
\mu_{m,n}(A) &= \mu(A \cap E_m \cap F_n) \\
\nu_{m,n}(A) &= \nu(A \cap E_m \cap F_n)
\end{align*}
\]

By the previous step, we have

\[ d\nu_{m,n} = d\lambda_{m,n} + f_{m,n} d\mu_{m,n} \] with $\lambda_{m,n} \perp \mu_{m,n}$

It now suffices to set

\[
\lambda = \sum_{m,n=1}^{\infty} \lambda_{m,n} \quad f = \sum_{m,n=1}^{\infty} f_{m,n} \chi_{E_m \cap F_n}
\]

Part 4: Proof in the general case:

Just apply the previous case separately to $\nu_+$ and $\nu_-$. \qed

Remark 7.6.

Here is some notation and terminology.

(a) The decomposition $\nu = \lambda + \rho$ with $\lambda \perp \mu$ and $\rho \ll \mu$ is called the Lebesgue decomposition of $\nu$ with respect to $\mu$.

(b) If $\nu \ll \mu$, then $d\nu = f \, d\mu$ for some $f$. The is called the Radon-Nikodym theorem and we write $f = \frac{d\nu}{d\mu}$.