Lecture Notes on Measure Theory and Integration

7 — The Radon-Nikodym Theorem

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7 The Radon-Nikodym Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space. One easy way to build another measure from μ is to take a measurable nonnegative function f and set

$$\nu(E) = \int_E f(x)\, d\mu(x) \qquad \text{for all } E \in \mathcal{M}$$

In this case, one usually writes $d\nu = f\, d\mu$.

Now suppose that someone gives you two measures, μ and ν, on $\mathcal{M}$. How can you tell if $d\nu = f\, d\mu$ for some f? There is one obvious necessary condition. If $d\nu = f\, d\mu$, for some function f (even if you don't know what it is), and if $E \in \mathcal{M}$ is a null set for μ (i.e. $\mu(E) = 0$), then

$$\nu(E) = \int_E f(x)\, d\mu(x) = \int \overbrace{f\chi_E}^{=0 \text{ a.e.}}\, d\mu = \int 0\, d\mu = 0$$

This property is given a name.

Definition 7.1. Let ν be a signed measure on $(X, \mathcal{M})$ and μ be a (positive) measure on $(X, \mathcal{M})$. Then ν is said to be **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if

$$E \in \mathcal{M},\ \mu(E) = 0 \implies \nu(E) = 0$$

In this part of the course we prove the Radon-Nikodym theorem, which says that, for σ-finite measures, absolute continuity is also a sufficient condition for ν to be of the form $d\nu = f\, d\mu$. Here is some prep.

Proposition 7.2. *Let ν be a finite signed measure (i.e. ν_+ and ν_- are both finite) and μ a (positive) measure on $(X, \mathcal{M})$. Then*

$$\nu \ll \mu \iff \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } E \in \mathcal{M},\ \mu(E) < \delta \implies |\nu(E)| < \varepsilon$$

Proof.
Part 1: Reduction to the case that ν is positive:

Use $|\nu|$ to denote the measure $\nu_+ + \nu_-$. Let $X = P \cup N$ be a Hahn decomposition (of Theorem 6.5) for ν. So

$$\nu_+(E) = \nu(E \cap P) \qquad \nu_-(E) = -\nu(E \cap N)$$

For the left hand side of the statement of the theorem

$$\begin{aligned}
\nu \ll \mu &\iff \Big(\overbrace{\mu(E) = 0}^{\iff \mu(E\cap P)+\mu(E\cap N)=0} \implies \nu(E) = 0 \Big) \\
&\iff \Big(\mu(E) = 0 \implies \nu_+(E) = \nu_-(E) = 0 \Big) \\
&\iff \Big(\mu(E) = 0 \implies |\nu|(E) = 0 \Big) \\
&\iff |\nu| \ll \mu
\end{aligned}$$

Call the statement on the right hand side of the theorem S_ν. That is, S_ν is "$\forall \varepsilon > 0\ \exists \delta > 0$ such that $E \in \mathcal{M},\ \mu(E) < \delta \implies |\nu(E)| < \varepsilon$". We now show that S_ν is true if and only if $S_{|\nu|}$ is true.

$$\begin{aligned}
S_\nu &\implies \Big(\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } E \in \mathcal{M},\ \mu(E) < \delta \implies |\nu(E)| < \tfrac{\varepsilon}{2} \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(just replaced } \varepsilon \text{ by } \tfrac{\varepsilon}{2}) \\
&\implies \Big(\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } E \in \mathcal{M},\ \mu(E) < \delta \implies |\nu(E \cap N)|, \nu(E \cap P) < \tfrac{\varepsilon}{2} \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } \mu(E) < \delta \text{ implies } \mu(E \cap N), \mu(E \cap P) < \delta) \\
&\implies \Big(\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } E \in \mathcal{M},\ \mu(E) < \delta \implies |\nu|(E) < \varepsilon \Big) \\
&\implies S_{|\nu|}
\end{aligned}$$

and

$$\begin{aligned}
S_{|\nu|} &\implies \Big(\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } E \in \mathcal{M},\ \mu(E) < \delta \implies |\nu|(E) < \varepsilon \Big) \\
&\implies \Big(\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } E \in \mathcal{M},\ \mu(E) < \delta \implies |\nu(E)| < \varepsilon \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } |\nu(E)| \le |\nu|(E)) \\
&\implies S_\nu
\end{aligned}$$

Part 2: Proof of $\Longleftarrow$ in the case that ν is positive:

We are assuming that S_ν is true and have to prove that $\nu \ll \mu$. Let $E \in \mathcal{M}$. Then

$$\begin{aligned}
\mu(E) = 0 &\overset{S_\nu}{\implies} |\nu(E)| < \varepsilon \text{ for all } \varepsilon > 0 \\
&\implies \nu(E) = 0
\end{aligned}$$

so that $\nu \ll \mu$.

Part 3: Proof of $\implies$ in the case that ν is positive:

We are assuming that $\nu \ll \mu$ and have to prove that S_ν is true. Assume that S_ν is false. Then there is an $\varepsilon > 0$ and, for every $n \in \mathbb{N}$, there is an $E_n \in \mathcal{M}$ such that $\mu(E_n) < \frac{1}{2^n}$ and $\nu(E_n) \geq \varepsilon$. Set

$$F_k = \bigcup_{n=k}^{\infty} E_n \qquad F = \bigcap_{k=1}^{\infty} F_k$$

Then

$$\mu(F_k) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{2}{2^k} \implies \mu(F) = \lim_{k\to\infty} \mu(F_k) = 0$$

but

$$\nu(F_k) \geq \nu(E_k) \geq \varepsilon \overset{\nu \text{ finite!}}{\implies} \nu(F) = \lim_{k\to\infty} \nu(F_k) \geq \varepsilon$$

contradicting the assumption that $\nu \ll \mu$.

□

Corollary 7.3. *Let $f \in L^1(\mu)$. Then*

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } E \in \mathcal{M},\ \mu(E) < \delta \implies \left| \int_E f(x)\ d\mu(x) \right| < \varepsilon$$

Proof. Apply Proposition 7.2 with $\nu(E) = \int_E f(x)\ d\mu(x)$.

□

THEOREM 7.4 (The Radon-Nikodym Theorem)**.** *Let ν be a σ-finite signed measure and μ be a σ-finite (positive) measure on $(X, \mathcal{M})$. There exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that*

$$\lambda \perp \mu \qquad \rho \ll \mu \qquad \nu = \lambda + \rho$$

Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f\, d\mu$. Any two such functions are equal almost everywhere.

First, we have a preliminary

Lemma 7.5. *Let μ, ν be finite (positive) measures on $(X, \mathcal{M})$. Then either $\nu \perp \mu$ or*

$$\exists\ \varepsilon > 0,\ E \in \mathcal{M} \text{ such that } \mu(E) > 0 \text{ and } \nu \geq \varepsilon\mu \text{ on } E$$

That is, $\mu(E) > 0$ and E is a positive set for $\nu - \varepsilon\mu$.

Proof. Let, for each $n \in \mathbb{N}$, $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$. Let

$$P = \bigcup_{n=1}^{\infty} P_n \qquad N = \bigcap_{n=1}^{\infty} N_n = P^c$$

Then

$$\begin{aligned}
&\text{For every } n \in \mathbb{N},\ N_n \text{ is a negative set for } \nu - \tfrac{1}{n}\mu \\
&\implies N \text{ is a negative set for } \nu - \tfrac{1}{n}\mu \text{ for all } n \in \mathbb{N} \\
&\implies \nu(N) - \tfrac{1}{n}\mu(N) \leq 0 \text{ for all } n \in \mathbb{N} \\
&\implies 0 \leq \nu(N) \leq \tfrac{1}{n}\mu(N) \text{ for all } n \in \mathbb{N} \\
&\implies \nu(N) = 0
\end{aligned}$$

If $\mu(P) = 0$, then $\nu \perp \mu$.
If $\mu(P) > 0$, then, by continuity from below, $\mu(P_n) > 0$ for some $n \in \mathbb{N}$, and then P_n is a positive set for $\nu - \frac{1}{n}\mu$.

□

Proof of Theorem 7.4.
Part 1: A little motivation in the case that μ, ν are finite and positive:

We want to guess a function f so that, if we set $d\rho = f d\mu$, then $\lambda = \nu - \rho$ is mutually singular with respect to μ. So let's suppose that we already know that

$$\begin{array}{llll}
\nu = \lambda + \rho & d\rho = f\, d\mu & \lambda \perp \mu & \\
X = L \cup M & L \cap M = \emptyset & \lambda(M) = 0 & \mu(L) = 0
\end{array}$$

for some (unknown) λ, ρ, L, M, and see what conditions (that do not mention λ, ρ, L, M — which we do not yet know) f has to obey.

- Since ν is positive and $\lambda(E) = \lambda(E \cap L) = \nu(E \cap L) \geq 0$,

$$\nu(E) = \lambda(E) + \int_E f\, d\mu \geq \int_E f\, d\mu \qquad \text{for all } E \in \mathcal{M}$$

and we'll use

$$\mathcal{F} = \left\{ \varphi : X \to [0, \infty],\ \text{measurable} \,\middle|\, \textstyle\int_E \varphi\, d\mu \leq \nu(E)\ \forall E \in \mathcal{M} \right\}$$

as the set of possible candidates for the function f.

- Think of $\nu(X)$, $\lambda(X) = \nu(L)$ and $\rho(X) = \nu(M)$ as being the "mass" of X, L and M, respectively. We will try to pick f to be the candidate $\varphi \in \mathcal{F}$ which puts as much mass, $\int_X \varphi \, d\mu$, as possible into $\rho(X)$. The remaining mass, which cannot be put into $\rho(X)$ because it is not in M, will go into $\lambda(X)$, i.e. into L.

Part 2: Proof of existence in the case that μ, ν are finite and positive:

The set $\mathcal{F}$ of candidates for the function f has the properties

- $\mathcal{F} \neq \emptyset$ since $\varphi = 0 \in \mathcal{F}$
- if $\varphi, \psi \in \mathcal{F}$, then $\zeta = \max\{\varphi, \psi\} \in \mathcal{F}$ since, writing $A = \big\{x \in X \big| \varphi(x) > \psi(x)\big\}$,

$$\begin{aligned}\int_E \zeta \, d\mu &= \int_{A \cap E} \varphi \, d\mu + \int_{E \setminus A} \psi \, d\mu \\ &\leq \nu(A \cap E) + \nu(E \setminus A) = \nu(E)\end{aligned}$$

Set

$$a = \sup \left\{ \int_X \varphi \, d\mu \,\middle|\, \varphi \in \mathcal{F} \right\} \leq \nu(X) < \infty$$

and choose $\big\{\varphi_n\big\}_{n \in \mathbb{N}} \subset \mathcal{F}$ such that $\lim_{n\to\infty} \int_X \varphi_n \, d\mu = a$. Then set

$$g_n(x) = \max\{\varphi_1(x), \cdots, \varphi_n(x)\} \qquad f(x) = \sup_n \varphi_n(x)$$

Then

- $g_n \in \mathcal{F}$, for all $n \in \mathbb{N}$
- $g_n(x)$ increases to $f(x)$, as $n \to \infty$, for all $\in X$ and
- $a \geq \int_X g_n \, d\mu \geq \int_X \varphi_n \, d\mu \overset{n\to\infty}{\longrightarrow} a$

which forces

$$\lim_{n\to\infty} \int_X g_n \, d\mu = a$$

By the monotone convergence theorem

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E g_n \, d\mu \leq \nu(E)$$

So

$$f \in \mathcal{F} \qquad \int_X f \, d\mu = a < \infty \qquad 0 \leq f < \infty \text{ a.e.}$$

Now we set

$$d\rho = f \, d\mu \qquad \lambda = \nu - \rho$$

Then

- $\nu = \lambda + \rho$
- $\rho \ll \mu$, ρ is positive
- λ is positive, since $f \in \mathcal{F}$ so that $\rho(E) = \int_E f\,d\mu \le \nu(E)$ for all $E \in \mathcal{M}$
- $\lambda \perp \mu$

  Proof: If not, then by the preliminary Lemma 7.5, there is an $\varepsilon > 0$ and an $E_0 \in \mathcal{M}$ with $\mu(E_0) > 0$ and $\lambda \ge \varepsilon\mu$ on E_0. Set $d\rho' = \varepsilon\chi_{E_0}\,d\mu$. Then $\rho' \le \lambda = \nu - \rho$, or $\rho + \rho' \le \nu$. That is

$$(f + \varepsilon\chi_{E_0})\,d\mu \le d\nu \implies f + \varepsilon\chi_{E_0} \in \mathcal{F} \text{ and } \int_X (f + \varepsilon\chi_{E_0})\,d\mu = a + \varepsilon\mu(E_0) > a$$

  which contradicts the definition of a.

Part 3: Proof of uniqueness in the case that μ, ν are finite and positive:

Suppose that

$$\begin{array}{llllll} \nu = \lambda + \rho & d\rho = f\,d\mu & X = E \cup F & E \cap F = \emptyset & \mu(E) = 0 & \lambda(F) = 0 \\ \nu = \lambda' + \rho' & d\rho' = f'\,d\mu & X = E' \cup F' & E' \cap F' = \emptyset & \mu(E') = 0 & \lambda'(F') = 0 \end{array}$$

We want to show that $\lambda = \lambda'$, $\rho = \rho'$ and $f = f'$ a.e.. We have $\lambda - \lambda' = \rho' - \rho$ and

- $\lambda - \lambda' \perp \mu$ since $\mu(E \cup E') = 0$ and $(\lambda - \lambda')(G) = 0$ for all $G \subset \overbrace{(E \cup E')^c}^{F \cap F'}$.
- $\rho' - \rho = (f' - f)\,d\mu \ll \mu$

which implies that, for all $S \in \mathcal{M}$,

$$\begin{aligned} (\lambda - \lambda')(S) &= (\lambda - \lambda')\big(S \cap (E \cup E')\big) \\ &= (\rho' - \rho)\big(S \cap (E \cup E')\big) \\ &= 0 \qquad \text{since } \mu\big(S \cap (E \cup E')\big) = 0 \\ \implies (\rho' - \rho)(S) &= 0 \end{aligned}$$

which implies that $\lambda = \lambda'$ and that $\rho = \rho'$. Finally, to show that $f' - f = 0$ a.e., set, for each $n \in \mathbb{N}$, $P_n = \left\{ x \;\middle|\; f'(x) \ge f(x) + 1/n \right\}$ and $N_n = \left\{ x \;\middle|\; f(x) \ge f'(x) + 1/n \right\}$.

Then

$$(\rho' - \rho)(P_n) = 0 \implies \int_{P_n} \overbrace{(f' - f)}^{\geq 1/n} \, d\mu = 0 \implies \mu(P_n) = 0$$

$$(\rho' - \rho)(N_n) = 0 \implies \int_{N_n} \overbrace{(f' - f)}^{\leq -1/n} \, d\mu = 0 \implies \mu(N_n) = 0$$

Part 4: Proof in the case that μ, ν are σ-finite and positive:

Write

$$X = \bigcup_{m=1}^{\infty} E_m \quad \text{with the } E_m\text{'s disjoint and each } \mu(E_m) < \infty$$

$$X = \bigcup_{n=1}^{\infty} F_n \quad \text{with the } F_n\text{'s disjoint and each } \nu(F_n) < \infty$$

Then

$$X = \bigcup_{m,n=1}^{\infty} E_m \cap F_n$$

with the $E_m \cap F_n$'s all disjoint and each $\mu(E_m \cap F_n), \nu(E_m \cap F_n) < \infty$. Set, for each $m, n \in \mathbb{N}$,

$$\mu_{m,n}(A) = \mu(A \cap E_m \cap F_n)$$
$$\nu_{m,n}(A) = \nu(A \cap E_m \cap F_n)$$

By the previous step, we have

$$d\nu_{m,n} = d\lambda_{m,n} + f_{m,n} \, d\mu_{m,n} \qquad \text{with } \lambda_{m,n} \perp \mu_{m,n}$$

It now suffices to set

$$\lambda = \sum_{m,n=1}^{\infty} \lambda_{m,n} \qquad f = \sum_{m,n=1}^{\infty} f_{m,n} \, \chi_{E_m \cap F_n}$$

Part 4: Proof in the general case:

Just apply the previous case separately to ν_+ and ν_-.

$\square$

Remark 7.6.
Here is some notation and terminology.

(a) The decomposition $\nu = \lambda + \rho$ with $\lambda \perp \mu$ and $\rho \ll \mu$ is called the Lebesgue decomposition of ν with respect to μ.

(b) If $\nu \ll \mu$, then $d\nu = f\, d\mu$ for some f. The is called the Radon-Nikodym theorem and we write $f = \frac{d\nu}{d\mu}$.