5 Product Measures and Fubini-Tonelli

Our goal in this section is to prove the Fubini-Tonelli theorem, which says that, under appropriate hypotheses,

$$\int f(x, y) \, d\mu \times \nu(x, y) = \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(x)$$

$$= \int \left[ \int f(x, y) \, d\mu(x) \right] \, d\nu(y)$$

5.1 Product Measures

Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces. Our first task is to define the product.

**Definition 5.1 (Product Measure).** Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces.

(a) Define the set of finite disjoint unions of measurable rectangles in \(X \times Y\) to be

\[
\mathcal{R} = \left\{ \bigcup_{j=1}^{n} A_j \times B_j \mid n \in \mathbb{N}, \ A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ (A_j \times B_j) \cap (A_k \times B_k) = \emptyset, \ \text{for all } 1 \leq j, k \leq n \text{ with } j \neq k \right\}
\]

\(\mathcal{R}\) is nonempty and closed under complements and finite unions and so is an algebra.

(b) Define \(\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R})\) to be the \(\sigma\)-algebra generated by \(\mathcal{R}\).

(c) Define \(\pi : \mathcal{R} \to [0, \infty]\) by

\[
\pi \left( \bigcup_{j=1}^{n} A_j \times B_j \right) = \sum_{j=1}^{n} \mu(A_j)\nu(B_j)
\]

for all \(n \in \mathbb{N}\) and all \(A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ 1 \leq j \leq n\) with \((A_j \times B_j) \cap (A_k \times B_k) = \emptyset\) for all \(j \neq k\). In this definition, we use the convention that \(0 \times \infty = 0\). \(\pi\) is a well-defined premeasure.

(d) Let \(\pi^*\) be the outer measure generated by \(\pi\). By Theorem 2.36,

\[
\mu \times \nu = \pi^* \upharpoonright \mathcal{M} \otimes \mathcal{N}
\]
is a measure which extends $\pi$. If $\mu$ and $\nu$ are $\sigma$-finite, then $\mu \times \nu$ is $\sigma$-finite and is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that

$$
\mu \times \nu(A \times B) = \mu(A) \nu(B) \quad \forall \ A \in \mathcal{M}, \ B \in \mathcal{N}
$$

**Remark 5.2.** (a) Any finite union of measurable rectangles can also be expressed as a finite disjoint union of measurable rectangles. So

$$
\mathcal{R} = \left\{ \bigcup_{j=1}^{n} A_j \times B_j \mid n \in \mathbb{N}, \ A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ \text{for all } 1 \leq j, k \leq n \right\}
$$

too. As $(A \times B)^c = (X \times B^c) \cup (A^c \times B)$, $\mathcal{R}$ written in this form is obviously an algebra.

(b) That $\pi$ is a well–defined premeasure (in part (c) of the definition) is a consequence of the observation that, if

$$
\bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k = \bigcup_{j=1}^{\infty} A_j \times B_j
$$

is a countable disjoint union of measurable rectangles, then

$$
\sum_{k=1}^{n} \chi_{\tilde{A}_k}(x) \chi_{\tilde{B}_k}(y) = \sum_{k=1}^{n} \chi_{\tilde{A}_k \times \tilde{B}_k}(x, y) = \chi_{\bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k}(x, y) = \chi_{\bigcup_{j=1}^{\infty} A_j \times B_j}(x, y)
$$

$$
= \sum_{j=1}^{\infty} \chi_{A_j \times B_j}(x, y) = \sum_{j=1}^{\infty} \chi_{A_j}(x) \chi_{B_j}(y)
$$

So integrating $d\mu(x)$ gives

$$
\sum_{k=1}^{n} \mu(\tilde{A}_k) \chi_{\tilde{B}_k}(y) = \sum_{j=1}^{\infty} \mu(A_j) \chi_{B_j}(y)
$$

by the monotone convergence theorem. Then integrating $d\nu(y)$ gives

$$
\sum_{k=1}^{n} \mu(\tilde{A}_k) \nu(\tilde{B}_k) = \sum_{j=1}^{\infty} \mu(A_j) \nu(B_j)
$$

$$
\pi\left( \bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k \right)
$$

again by the Monotone Convergence Theorem.
(c) For the $\sigma$-finite statement in part (d) of the definition, observe that if
\[
\{A_j\}_{j \in \mathbb{N}} \subset \mathcal{M}, \ \{B_k\}_{k \in \mathbb{N}} \subset \mathcal{N}, \ X = \bigcup_{j \in \mathbb{N}} A_j, \ Y = \bigcup_{k \in \mathbb{N}} B_k, \ \mu(A_j) < \infty, \ \nu(B_k) < \infty
\]
then
\[
\{A_j \times B_k\}_{j,k \in \mathbb{N}} \subset \mathcal{M} \otimes \mathcal{N}, \ X \times Y = \bigcup_{j,k \in \mathbb{N}} A_j \times B_k
\]
and
\[
\mu \times \nu(A_j \times B_k) = \mu(A_j) \nu(B_k) < \infty
\]

**Proposition 5.3** (Tensor product of Borel $\sigma$-algebras). Let $X$ and $Y$ be separable\(^1\) metric spaces with metrics $d_X$ and $d_Y$ respectively. Then $X \times Y$ is a metric space with metric $D((x, y), (x', y')) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}$ and $\mathcal{B}_{X \times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. (By definition, a metric space is separable if and only if it contains a countable dense subset. For example, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, so that $\mathbb{R}$ is separable. Applying this proposition to $X = Y = \mathbb{R}$ gives that $\mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R} = \mathcal{B}_{\mathbb{R}^2}$.)

*Proof.* Problem Set 9, #5. \(\square\)

**Proposition 5.4** (Slices — sets). Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, $x \in X$ and $y \in Y$.

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$, then
\[
E_x = \{ y' \in Y \mid (x, y') \in E \} \in \mathcal{N}
\]
\[
E^y = \{ x' \in X \mid (x', y) \in E \} \in \mathcal{M}
\]

\(1\)For a counterexample in the nonseparable case, see Exercise 29 on page 231 of Folland.
(b) If $f : X \times Y \to \mathbb{R}$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ is $\mathcal{N}$-measurable and the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x, y)$ is $\mathcal{M}$-measurable.

**Proof.** (a) Let

$$\mathcal{P} = \{ \text{ E } \subset X \times Y \mid E_x \in \mathcal{N} \text{ for all } x \in X, \ E^y \in \mathcal{M} \text{ for all } y \in Y \}$$

Then

- $A \times B \in \mathcal{P}$ for all $A \in \mathcal{M}$ and $B \in \mathcal{N}$, since
  \[
  (A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \quad (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases}
  \]

- $\mathcal{P}$ is closed under complements since
  \[
  \{ y \in Y \mid (x, y) \notin E \} = (E^c)_x = (E_x)^c = \{ y \in Y \mid (x, y) \in E \}^c \in \mathcal{N} \\
  (E^c)^y = (E^y)^c \in \mathcal{M}
  \]

- $\mathcal{P}$ is closed under countable unions since
  \[
  (\bigcup_n E_n)_x = \bigcup_n (E_n)_x = \{ y \in Y \mid (x, y) \in E_n \text{ for some } n \} \in \mathcal{N} \\
  (\bigcup_n E_n)^y = \bigcup_n (E_n)^y \in \mathcal{M}
  \]

So $\mathcal{P}$ is a $\sigma$-algebra which contains $\mathcal{R}$, and hence contains $\mathcal{M}(\mathcal{R}) = \mathcal{M} \otimes \mathcal{N}$.

(b) Let $B \in \mathcal{B}_\mathbb{R}$. As $f$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, $f^{-1}(B) \in \mathcal{M} \otimes \mathcal{N}$. So

\[
 f^{-1}_x(B) = \{ y \in Y \mid f_x(y) = f(x, y) \in B \} = \bigcup_{y \in \mathcal{N}} f^{-1}_x(B)_y \\
 (f^y)^{-1}(B) = \{ x \in X \mid f^y(x) = f(x, y) \in B \} = \bigcup_{x \in \mathcal{M}} f^{-1}(B)^y
\]

$\square$
5.2 Technical Aside — Monotone Classes

Definition 5.5. Let $X$ be a nonempty set. A collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of $X$ is called a **monotone class** if
- it is closed under countable increasing unions (that is, if \( \{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C} \) and $E_1 \subset E_2 \subset E_3 \subset \cdots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$) and
- it is closed under countable decreasing intersections (that is, if \( \{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C} \) and $E_1 \supset E_2 \supset E_3 \supset \cdots$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$).

Remark 5.6.
(a) Monotone classes are closely related to $\sigma$–algebras. In fact, for us, their only use will be to help verify that a certain collection of subsets is a $\sigma$–algebra.
(b) Every $\sigma$–algebra is a monotone class, because $\sigma$–algebras are closed under arbitrary countable unions and intersections.
(c) If, for every index $i$ in some index set $\mathcal{I}$, $\mathcal{C}_i$ is a monotone class, then $\bigcap_{i \in \mathcal{I}} \mathcal{C}_i$ is also a monotone class. In particular, for any $\mathcal{E} \subset \mathcal{P}(X)$, the collection

\[
\mathcal{C}(\mathcal{E}) = \bigcap_{\mathcal{C} \text{ monotone class}} \mathcal{C}
\]

is a monotone class, called the monotone class generated by $\mathcal{E}$. It is the smallest monotone class that contains $\mathcal{E}$. So if $\mathcal{C}$ is any monotone class that contains $\mathcal{E}$, then $\mathcal{C}(\mathcal{E}) \subset \mathcal{C}$.

Lemma 5.7. Let $X$ be a nonempty set. If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, then

\[
\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})
\]

That is, the monotone class generated by $\mathcal{A}$ is the same as the $\sigma$–algebra generated by $\mathcal{A}$.

Proof.
- $\mathcal{C}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$: \ $\mathcal{M}(\mathcal{A})$ is a $\sigma$–algebra, and hence a monotone class, that contains $\mathcal{A}$. So, this follows by part (c) of Remark 5.6.
- $\mathcal{M}(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$: It suffices to prove that $\mathcal{C}(\mathcal{A})$ is a $\sigma$–algebra, because then we will know that $\mathcal{C}(\mathcal{A})$ is a $\sigma$–algebra containing $\mathcal{A}$ and hence $\mathcal{M}(\mathcal{A})$, which is the smallest $\sigma$–algebra containing $\mathcal{A}$. 

By Problem Set 1, # 6, any algebra that is closed under countable increasing unions is a \(\sigma\)–algebra. So it suffices to prove that \(\mathcal{C}(\mathcal{A})\) is an algebra (i.e. that \(\mathcal{C}(\mathcal{A})\) is nonempty and closed under complements and finite intersections). So it suffices to prove

\[
E, F \in \mathcal{C}(\mathcal{A}) \implies E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A})
\]

(*)

(since \(X\) is automatically in \(\mathcal{A}\), which is an algebra, and hence is automatically in \(\mathcal{C}(\mathcal{A})\) and is an allowed choice for \(E\)). Let \(E \in \mathcal{C}(\mathcal{A})\) and define

\[
\mathcal{D}(E) = \{ \, F \in \mathcal{C}(\mathcal{A}) \mid E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(\mathcal{A}) \, \}
\]

We wish to show that \(\mathcal{C}(\mathcal{A}) \subseteq \mathcal{D}(E)\). To do so, it suffices to show that \(\mathcal{D}(E)\) is a monotone class that contains \(\mathcal{A}\). We first prove some properties of \(\mathcal{D}(E)\).

(a) \(\emptyset, E \in \mathcal{D}(E)\).

(b) For \(E, F \in \mathcal{C}(\mathcal{A}), F \in \mathcal{D}(E) \iff E \in \mathcal{D}(F)\).

(c) \(\mathcal{D}(E)\) is closed under countable increasing unions. To see this, let the sequence \(\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{D}(E)\) obey \(F_1 \subset F_2 \subset F_3 \subset \cdots\) and set \(F = \bigcup_{n=1}^\infty F_n\). Then

\[
\begin{align*}
\{ E \setminus F_n = E \cap F_n^c \}_{n \in \mathbb{N}} & \subseteq \mathcal{C}(\mathcal{A}) \text{ is decreasing}, \\
\{ F_n \setminus E = F_n \cap E^c \}_{n \in \mathbb{N}} & \subseteq \mathcal{C}(\mathcal{A}) \text{ is increasing and} \\
\{ E \cap F_n \}_{n \in \mathbb{N}} & \subseteq \mathcal{C}(\mathcal{A}) \text{ is increasing},
\end{align*}
\]

so that

\[
E \setminus F = E \cap \Big( \bigcup_{n=1}^\infty F_n \Big)^c = E \cap \Big( \bigcap_{n=1}^\infty F_n^c \Big) = \bigcap_{n=1}^\infty (E \cap F_n^c) \in \mathcal{C}(\mathcal{A})
\]

\[
F \setminus E = \Big( \bigcup_{n=1}^\infty F_n \Big) \cap E^c = \bigcup_{n=1}^\infty (F_n \cap E^c) = \bigcup_{n=1}^\infty (F_n \setminus E) \in \mathcal{C}(\mathcal{A})
\]

\[
E \cap F = E \cap \Big( \bigcup_{n=1}^\infty F_n \Big) = \bigcup_{n=1}^\infty (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\]

since \(\mathcal{C}(\mathcal{A})\) is closed under countable decreasing intersections and countable increasing unions. So \(F \in \mathcal{D}(E)\).

(d) \(\mathcal{D}(E)\) is closed under countable decreasing intersections. To see this, let \(\{F_n\}_{n \in \mathbb{N}} \subseteq \mathcal{D}(E)\) obey \(F_1 \supset F_2 \supset F_3 \supset \cdots\) and set \(F = \bigcap_{n=1}^\infty F_n\). As in
part (c)
\[
E \setminus F = E \cap \left( \bigcap_{n=1}^{\infty} F_n \right)^c = E \cap \left( \bigcup_{n=1}^{\infty} F_n^c \right) = \bigcup_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(\mathcal{A})
\]
\[
F \setminus E = \left( \bigcap_{n=1}^{\infty} F_n \right) \cap E^c = \bigcap_{n=1}^{\infty} (F_n \cap E^c) = \bigcap_{n=1}^{\infty} (F_n \setminus E) \in \mathcal{C}(\mathcal{A})
\]
\[
E \cap F = E \cap \left( \bigcap_{n=1}^{\infty} F_n \right) = \bigcap_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(\mathcal{A})
\]

So \( F \in \mathcal{D}(E) \).

We are now ready to prove \((*)\), or equivalently, that \( \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E) \) for all \( E \in \mathcal{C}(\mathcal{A}) \). Let \( E \in \mathcal{C}(\mathcal{A}) \). By properties (c) and (d), \( \mathcal{D}(E) \) is a monotone class, so it suffices to prove that \( \mathcal{A} \subset \mathcal{D}(E) \). But

\[
F \in \mathcal{A} \implies \mathcal{A} \subset \mathcal{D}(F) \quad \text{by the definition of } \mathcal{D}(F), \text{ since } \mathcal{A} \text{ is an algebra}
\]
\[
\implies \mathcal{C}(\mathcal{A}) \subset \mathcal{D}(F) \quad \text{since } \mathcal{D}(\mathcal{F}) \text{ is a monotone class}
\]
\[
\implies E \in \mathcal{D}(F) \quad \text{since } E \in \mathcal{C}(\mathcal{A})
\]
\[
\implies F \in \mathcal{D}(E) \quad \text{by property (b)}
\]

\[\Box\]

5.3 Fubini-Tonelli

**Proposition 5.8** (Slices — measure). Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be \(\sigma\)-finite measure spaces and let \( E \in \mathcal{M} \otimes \mathcal{N} \). Then the function \( f : X \to [0, \infty] \) defined by \( f(x) = \nu(E_x) \) is \(\mathcal{M}\)-measurable and the function \( g : Y \to [0, \infty] \) defined by \( g(y) = \mu(E^y) \) is \(\mathcal{N}\)-measurable and

\[
\mu \times \nu(E) = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y)
\]

**Proof.**

*Case 1: \( \mu, \nu \) finite:*

Set

\[
\mathcal{C} = \{ \ E \in \mathcal{M} \otimes \mathcal{N} \mid \text{conclusions of the theorem are true} \}
\]

Then

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\( A \in \mathcal{M}, \; B \in \mathcal{N} \implies A \times B \in \mathcal{C} \) since

\[
\nu((A \times B)_x) = \nu\left( \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \right) = \chi_A(x) \nu(B)
\]

\[
\mu((A \times B)_y) = \chi_B(y) \mu(A)
\]

\[
\int \nu((A \times B)_x) \, d\mu(x) = \mu(A) \nu(B) = \mu \times \nu(A \times B)
\]

\[
\int \mu((A \times B)_y) \, d\nu(x) = \mu(A) \nu(B) = \mu \times \nu(A \times B)
\]

\( \mathcal{C} \) is closed under finite disjoint unions since

\[
\nu\left( \bigcup_{\text{disjoint}} E_x \right) = \nu\left( \bigcup_{\text{disjoint}} F_x \right) = \nu(E_x) + \nu(F_x)
\]

Similarly for \( \mu((E \cup F)_y) \). So \( \mathcal{R} \subset \mathcal{C} \).

\( \mathcal{C} \) is closed under countable increasing unions:

If \( E_1 \subset E_2 \subset E_3 \subset \cdots \) are all in \( \mathcal{C} \) and \( E = \bigcup_{n \in \mathbb{N}} E_n \) then

\[
\{ f_n(x) = \nu((E_n)_x) \}_{n \in \mathbb{N}} \text{ increases pointwise to } f(x) = \nu(E_x)
\]

by continuity from below, so \( f \) is measurable and by the monotone convergence theorem

\[
\int \nu(E_x) \, d\mu(x) = \lim_{n \to \infty} \int \nu((E_n)_x) \, d\mu(x) = \lim_{n \to \infty} \mu \times \nu(E_n) = \mu \times \nu(E)
\]

by continuity from below. Similarly for \( \int \mu(E_y) \, d\nu(y) \).

\( \mathcal{C} \) is closed under countable decreasing intersections:

If \( E_1 \supset E_2 \supset E_3 \supset \cdots \) are all in \( \mathcal{C} \) and \( E = \bigcap_{n \in \mathbb{N}} E_n \) then

\[
\{ f_n(x) = \nu((E_n)_x) \}_{n \in \mathbb{N}} \text{ decreases pointwise to } f(x) = \nu(E_x)
\]

by continuity from above. (We used \( \nu(E_1) < \infty \) here.) So \( f \) is measurable and \( 0 \leq f_n(x) \leq f_1(x) \in L^1(X, \mathcal{M}, \mu) \) and, by the dominated
convergence theorem,
\[ \int \nu(E_x) \, d\mu(x) = \int f(x) \, d\mu(x) = \lim_{n \to \infty} \int f_n(x) \, d\mu(x) \]
\[ = \lim_{n \to \infty} \int \nu((E_n)_x) \, d\mu(x) = \lim_{n \to \infty} \mu \times \nu(E_n) \]
\[ = \mu \times \nu(E) \]

by continuity from above. Similarly for \( \int \mu(E^y) \, d\nu(y) \).

So \( C \) is a monotone class that contains \( \mathcal{R} \) and hence contains \( \mathcal{M} \otimes \mathcal{N} \).

Case 2: \( \mu, \nu \) \( \sigma \)-finite:

We can write \( X \times Y \) as a countable increasing union of rectangles \( \{X_n \times Y_n\}_{n \in \mathbb{N}} \) of finite measure. Then, if \( E \in \mathcal{M} \otimes \mathcal{N} \), we can apply the previous case to \( E \cap (X_n \times Y_n) \).

\[ \mu \times \nu(E \cap (X_n \times Y_n)) = \int \mu\left( (E \cap (X_n \times Y_n))^y \right) \, d\nu(y) \]
\[ = \int \mu(E^y \cap X_n) \, \chi_{Y_n}(y) \, d\nu(y) \]

In the limit as \( n \to \infty \),
- the left hand side converges to \( \mu \times \nu(E) \) by continuity from below and
- the right hand side converges to \( \int \mu(E^y) \, d\nu(y) \) by the monotone convergence theorem.

\[ \square \]

**THEOREM 5.9** (Fubini\(^2\)-Tonelli\(^3\)). Let \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) be \( \sigma \)-finite measure spaces.

(a) (Tonelli) If the function \( f : X \times Y \to [0, \infty] \) is \( \mathcal{M} \otimes \mathcal{N} \)-measurable, then
- the function \( g : X \to [0, \infty] \) defined by \( g(x) = \int f(x, y) \, d\nu(y) \) is \( \mathcal{M} \)-measurable, and
- the function \( h : Y \to [0, \infty] \) defined by \( h(y) = \int f(x, y) \, d\mu(x) \) is \( \mathcal{N} \)-measurable

\(^2\)Guido Fubini (1879-1943) was an Italian mathematician  
\(^3\)Leonida Tonelli (1885-1946) was an Italian mathematician
and

\[ \int f(x, y) \, d\mu \times \nu(x, y) = \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(x) = \int \left[ \int f(x, y) \, d\mu(x) \right] \, d\nu(y) \]

(b) (Fubini) If \( f \in L^1(\mu \times \nu) \) then

- the function \( f_x : Y \to \mathbb{R} \) defined by \( f_x(y) = f(x, y) \) is in \( L^1(\nu) \) for almost all \( x \in X \),
- \( g(x) = \int f(x, y) \, d\nu(y) \in L^1(\mu) \)
- the function \( f^y : X \to \mathbb{R} \) defined by \( f^y(x) = f(x, y) \) is in \( L^1(\mu) \) for almost all \( y \in Y \),

and

\[ \int f(x, y) \, d\mu \times \nu(x, y) = \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(x) = \int \left[ \int f(x, y) \, d\mu(x) \right] \, d\nu(y) \]

Proof.

(a) (Tonelli)

Case 1: \( f = \chi_E \) with \( E \in \mathcal{M} \otimes \mathcal{N} \):

This is Proposition 5.8.

Case 2: \( f \geq 0 \) simple:

This follows from Case 1 by linearity.

Case 3: \( f \in L^+(X \times Y, \mathcal{M} \otimes \mathcal{N}) \):

Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of nonnegative simple functions that increase pointwise to \( f \). For example, we could take

\[ f_n = \sum_{m=0}^{4^n-1} \frac{m}{2^n} \chi_{f^{-1}(I_{m,n})} + 2^n \chi_{f^{-1}([2^n, \infty))}(x) \quad \text{where} \quad I_{m,n} = f^{-1}(\left[ \frac{m}{2^n}, \frac{m+1}{2^n} \right)) \]

Then, by the monotone convergence theorem

\[ \int f_n(x, y) \, d\mu \times \nu(x, y) = \int \left[ \int f_n(x, y) \, d\nu(y) \right] \, d\mu(x) \]
and similarly for the other order.

(b) (Fubini)

Write

\[ h(y) = \int f^y(x) \, d\mu(x) = \int f(x, y) \, d\mu(x) \]
\[ g(x) = \int f_x(y) \, d\nu(y) = \int f(x, y) \, d\nu(y) \]

By Tonelli,

\[ f(x, y) \in L^1 \implies \int |f(x, y)| \, d\mu \times \nu < \infty \]
\[ \implies \int \left[ \int |f(x, y)| \, d\mu(x) \right] \, d\nu(y) < \infty \]
\[ \implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } |h(y)| \leq \int |f(x, y)| \, d\mu(x) \in L^1 \]
\[ \implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } h(y) \in L^1(\nu) \]

Similarly

\[ f_x \in L^1(\nu) \text{ a.e. } x \text{ and } g(x) \in L^1(\mu) \]

Now just apply Tonelli to the positive and negative parts of \( f \).