Lecture Notes on Measure Theory and Integration
5 — Product Measures and the Fubini-Tonelli Theorem

Joel Feldman

University of British Columbia

November 15, 2019
5 Product Measures and Fubini-Tonelli

Our goal in this section is to prove the Fubini-Tonelli theorem\(^1\), which says that, under appropriate hypotheses,

\[
\int_{X \times Y} f(x, y) \, d\mu \times \nu(x, y) = \int_X \left[ \int_Y f(x, y) \, d\nu(y) \right] d\mu(x)
\]

\[
= \int_Y \left[ \int_X f(x, y) \, d\mu(x) \right] d\nu(y)
\]

5.1 Product Measures

Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces. Our first task is to define the product.

**Definition 5.1 (Product Measure).** Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces.

(a) Define the set of finite disjoint unions of measurable rectangles in \(X \times Y\) to be

\[
\mathcal{R} = \left\{ \bigcup_{j=1}^n A_j \times B_j \mid n \in \mathbb{N}, \ A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ (A_j \times B_j) \cap (A_k \times B_k) = \emptyset, \right. \]

for all \(1 \leq j, k \leq n\) with \(j \neq k\)

\[
\mathcal{R} \text{ is nonempty. We will shortly show that it is closed under complements and finite unions and so is an algebra.}
\]

(b) Define \(\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R})\) to be the \(\sigma\)-algebra generated by \(\mathcal{R}\).

(c) Define \(\pi : \mathcal{R} \rightarrow [0, \infty]\) by

\[
\pi \left( \bigcup_{j=1}^n A_j \times B_j \right) = \sum_{j=1}^n \mu(A_j) \nu(B_j)
\]

for all \(n \in \mathbb{N}\) and all \(A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ 1 \leq j \leq n\) with \((A_j \times B_j) \cap (A_k \times B_k) = \emptyset\) for all \(j \neq k\). In this definition, we use the convention that \(0 \times \infty = 0\). We will shortly show that \(\pi\) is a well-defined premeasure.

\(^1\)The special case of this theorem, for continuous functions on rectangles, was known to Euler in the 18\(^{th}\) century. Lebesgue extended this to bounded measurable functions in 1904. Fubini’s version was 1907, and Tonelli’s version was 1909.
(d) Let $\pi^*$ be the outer measure generated by $\pi$. By Theorem 2.36,

$$\mu \times \nu = \pi^* \upharpoonright \mathcal{M} \otimes \mathcal{N}$$

is a measure which extends $\pi$. We will shortly show that if $\mu$ and $\nu$ are $\sigma$-finite, then $\mu \times \nu$ is $\sigma$-finite. Then it is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that

$$\mu \times \nu(A \times B) = \mu(A) \nu(B) \ \forall \ A \in \mathcal{M}, \ B \in \mathcal{N}$$

**Remark 5.2.** (a) Any finite union of measurable rectangles can also be expressed as a finite disjoint union of measurable rectangles.

![Diagram](image)

So

$$\mathcal{R} = \left\{ \bigcup_{j=1}^{n} A_j \times B_j \ \big| \ n \in \mathbb{N}, \ A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ \text{for all } 1 \leq j, k \leq n \right\}$$

too. As $(A \times B)^c = (X \times B^c) \cup (A^c \times B)$, $\mathcal{R}$ written in this form is obviously an algebra.

(b) That $\pi$ is a well–defined premeasure (in part (c) of the definition) is a consequence of the observation that, if

$$\bigcup_{j=1}^{\infty} A_j \times B_j = \bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k$$

are disjoint unions of measurable rectangles, then

$$\sum_{k=1}^{n} \chi_{\tilde{A}_k}(x)\chi_{\tilde{B}_k}(y) = \sum_{k=1}^{n} \chi_{\tilde{A}_k \times \tilde{B}_k}(x, y) = \chi_{\bigcup_k \tilde{A}_k \times \tilde{B}_k}(x, y) = \chi_{\bigcup_j A_j \times B_j}(x, y)$$

$$= \sum_{j=1}^{\infty} \chi_{A_j \times B_j}(x, y) = \sum_{j=1}^{\infty} \chi_{A_j}(x) \chi_{B_j}(y)$$

So integrating $d\mu(x)$ gives

$$\sum_{k=1}^{n} \mu(\tilde{A}_k) \chi_{\tilde{B}_k}(y) = \sum_{j=1}^{\infty} \mu(A_j) \chi_{B_j}(y)$$
by the monotone convergence theorem. Then integrating $d\nu(y)$ gives

$$\sum_{k=1}^{n} \mu(\tilde{A}_k) \nu(\tilde{B}_k) = \sum_{j=1}^{\infty} \mu(A_j) \nu(B_j) \pi(\bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k)$$

again by the monotone convergence theorem.

(c) For the $\sigma$-finite statement in part (d) of the definition, observe that if

$$\{A_j\}_{j \in \mathbb{N}} \subset \mathcal{M}, \ {B_k\}_{k \in \mathbb{N}} \subset \mathcal{N}, \ X = \bigcup_{j \in \mathbb{N}} A_j, \ Y = \bigcup_{k \in \mathbb{N}} B_k, \ \mu(A_j) < \infty, \ \nu(B_k) < \infty$$

then

$$\{A_j \times B_k\}_{j,k \in \mathbb{N}} \subset \mathcal{M} \otimes \mathcal{N}, \ X \times Y = \bigcup_{j,k \in \mathbb{N}} A_j \times B_k$$

and

$$\mu \times \nu(A_j \times B_k) = \mu(A_j) \nu(B_k) < \infty$$

**Proposition 5.3** (Tensor product of Borel $\sigma$-algebras). Let $X$ and $Y$ be separable$^2$ metric spaces with metrics $d_X$ and $d_Y$ respectively. Then $X \times Y$ is a metric space with metric $D((x,y),(x',y')) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$ and $\mathcal{B}_{X \times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. (By definition, a metric space is separable if and only if it contains a countable dense subset. For example, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, so that $\mathbb{R}$ is separable. Applying this proposition to $X = Y = \mathbb{R}$ gives that $\mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R} = \mathcal{B}_{\mathbb{R}^2}$.)

**Proof.** Problem Set 9, #5.

**Proposition 5.4** (Slices — sets). Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measure spaces, $x \in X$ and $y \in Y$.

(a) If $E \in \mathcal{M} \otimes \mathcal{N}$, then

$$E_x = \{ \ y' \in Y \mid (x,y') \in E \} \in \mathcal{N}$$

$$E^y = \{ \ x' \in X \mid (x',y) \in E \} \in \mathcal{M}$$

$^2$For a counterexample in the nonseparable case, see Exercise 29 on page 231 of Folland.
(b) If $f : X \times Y \to \mathbb{R}$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ is $\mathcal{N}$-measurable and the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x, y)$ is $\mathcal{M}$-measurable.

Proof. (a) Let

$$\mathcal{P} = \{ E \subset X \times Y \mid E_x \in \mathcal{N} \text{ for all } x \in X, \ E^y \in \mathcal{M} \text{ for all } y \in Y \}$$

Then
- $A \times B \in \mathcal{P}$ for all $A \in \mathcal{M}$ and $B \in \mathcal{N}$, since

$$\begin{align*}
(A \times B)_x &= \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \\
(A \times B)^y &= \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases}
\end{align*}$$

- $\mathcal{P}$ is closed under complements since, if $E \in \mathcal{P}$, then

$$\begin{align*}
(E^c)_x &= (E^c_x)^c \in \mathcal{N} \\
(E^c)^y &= (E^c_y)^c \in \mathcal{M}
\end{align*}$$
• $\mathcal{P}$ is closed under countable unions since, if $E_n \in \mathcal{P}$ for all $n \in \mathbb{N}$, then

$$(\bigcup_n E_n)_x = \bigcup_{E_n \in \mathcal{N}} (E_n)_x \in \mathcal{N}$$

$$(\bigcup_n E_n)_y = \bigcup_{E_n \in \mathcal{M}} (E_n)_y \in \mathcal{M}$$

So $\mathcal{P}$ is a $\sigma$-algebra which contains $\mathcal{R}$, and hence contains $\mathcal{M}(\mathcal{R}) = \mathcal{M} \otimes \mathcal{N}$.

(b) Let $B \in \mathcal{B}_R$. As $f$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, $f^{-1}(B) \in \mathcal{M} \otimes \mathcal{N}$. So

$$f_x^{-1}(B) = \left\{ y \in Y \mid f_x(y) = f(x, y) \in B \right\} = \bigcap_{E \in \mathcal{M} \otimes \mathcal{N}} f_x^{-1}(E)_x$$

$$f_y^{-1}(B) = \left\{ x \in X \mid f_y(x) = f(x, y) \in B \right\} = \bigcap_{E \in \mathcal{M} \otimes \mathcal{N}} f_y^{-1}(E)_y$$

5.2 Technical Aside — Monotone Classes

**Definition 5.5.** Let $X$ be a nonempty set. A collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of $X$ is called a **monotone class** if

- it is closed under countable increasing unions (that is, if $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ and $E_1 \subset E_2 \subset E_3 \subset \cdots$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$) and
- it is closed under countable decreasing intersections (that is, if $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$).

**Remark 5.6.**

(a) Monotone classes are closely related to $\sigma$–algebras. In fact, for us, their only use will be to help verify that a certain collection of subsets is a $\sigma$–algebra.

(b) Every $\sigma$–algebra is a monotone class, because $\sigma$–algebras are closed under arbitrary countable unions and intersections.

(c) If, for every index $i$ in some index set $\mathcal{I}$, $\mathcal{C}_i$ is a monotone class, then $\bigcap_{i \in \mathcal{I}} \mathcal{C}_i$ is also a monotone class. In particular, for any $\mathcal{E} \subset \mathcal{P}(X)$, the collection

$$\mathcal{C}(\mathcal{E}) = \bigcap_{\mathcal{C} \text{ monotone class}} \mathcal{C}$$
is a monotone class, called the monotone class generated by $E$. It is the smallest monotone class that contains $E$. So if $C$ is any monotone class that contains $E$, then $C(E) \subset C$.

**Lemma 5.7.** Let $X$ be a nonempty set. If $A \subset \mathcal{P}(X)$ is an algebra, then

$$C(A) = \mathcal{M}(A)$$

That is, the monotone class generated by $A$ is the same as the $\sigma$–algebra generated by $A$.

**Proof.**

- $C(A) \subset \mathcal{M}(A)$:
  $\mathcal{M}(A)$ is a $\sigma$–algebra, and hence a monotone class, that contains $A$. So, this follows by part (c) of Remark 5.6.

- $\mathcal{M}(A) \subset C(A)$:
  It suffices to prove that $C(A)$ is a $\sigma$–algebra, because then we will know that $C(A)$ is a $\sigma$–algebra containing $A$ and hence $\mathcal{M}(A)$, which is the smallest $\sigma$–algebra containing $A$.
  By Problem Set 1, # 6, any algebra that is closed under countable increasing unions is a $\sigma$–algebra. So it suffices to prove that $C(A)$ is an algebra (i.e. that $C(A)$ is nonempty and closed under complements and finite intersections). So it suffices to prove

  $$E, F \in C(A) \implies E \setminus F, F \setminus E, E \cap F \in C(A) \quad \text{\textcopyright{(*)}}$$

  (since $X$ is automatically in $A$, which is an algebra, and hence is automatically in $C(A)$ and is an allowed choice for $E$). Define, for each $E \in C(A)$,

  $$\mathcal{D}(E) = \{ F \in C(A) \mid E \setminus F, F \setminus E, E \cap F \in C(A) \}$$

  We wish to show that

  $$E \in C(A) \implies C(A) \subset \mathcal{D}(E)$$

  To do so, it suffices to show that $\mathcal{D}(E)$ is a monotone class that contains $A$.
  We first prove some properties of $\mathcal{D}(E)$.
  (a) $E \in C(A) \implies \emptyset, E \in \mathcal{D}(E)$.
  (b) For $E, F \in C(A)$,

  $$F \in \mathcal{D}(E) \iff E \setminus F, F \setminus E, E \cap F \in C(A)$$

  $$\iff E \in \mathcal{D}(F) = \{ F' \in C(A) \mid F \setminus F', F' \setminus F, F \cap F' \in C(A) \}$$

Product Measures and Fubini-Tonelli
D(E) is closed under countable increasing unions. To see this, let the sequence \( \{F_n\}_{n \in \mathbb{N}} \subset D(E) \) obey \( F_1 \subset F_2 \subset F_3 \subset \cdots \) and set \( F = \bigcup_{n=1}^{\infty} F_n \). Then

\[
\begin{align*}
\{E \setminus F_n = E \cap F_n^c\}_{n \in \mathbb{N}} & \subset C(A) \text{ is decreasing,} \\
\{F_n \setminus E = F_n \cap E^c\}_{n \in \mathbb{N}} & \subset C(A) \text{ is increasing and} \\
\{E \cap F_n\}_{n \in \mathbb{N}} & \subset C(A) \text{ is increasing,}
\end{align*}
\]

so that

\[
\begin{align*}
E \setminus F &= E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c = E \cap \left(\bigcap_{n=1}^{\infty} F_n^c\right) = \bigcap_{n=1}^{\infty} (E \cap F_n^c) \in C(A) \\
F \setminus E &= \left(\bigcup_{n=1}^{\infty} F_n\right) \cap E^c = \bigcup_{n=1}^{\infty} (F_n \cap E^c) = \bigcup_{n=1}^{\infty} (F_n \setminus E) \in C(A) \\
E \cap F &= E \cap \left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} (E \cap F_n) \in C(A)
\end{align*}
\]

since \( C(A) \) is closed under countable decreasing intersections and countable increasing unions. So \( F \in D(E) \).

(d) \( D(E) \) is closed under countable decreasing intersections. To see this, let \( \{F_n\}_{n \in \mathbb{N}} \subset D(E) \) obey \( F_1 \supset F_2 \supset F_3 \supset \cdots \) and set \( F = \bigcap_{n=1}^{\infty} F_n \). As in part (c)

\[
\begin{align*}
E \setminus F &= E \cap \left(\bigcap_{n=1}^{\infty} F_n\right)^c = E \cap \left(\bigcup_{n=1}^{\infty} F_n^c\right) = \bigcup_{n=1}^{\infty} (E \cap F_n^c) \in C(A) \\
F \setminus E &= \left(\bigcap_{n=1}^{\infty} F_n\right) \cap E^c = \bigcap_{n=1}^{\infty} (F_n \cap E^c) = \bigcap_{n=1}^{\infty} (F_n \setminus E) \in C(A) \\
E \cap F &= E \cap \left(\bigcap_{n=1}^{\infty} F_n\right) = \bigcap_{n=1}^{\infty} (E \cap F_n) \in C(A)
\end{align*}
\]

So \( F \in D(E) \).

We are now ready to prove (*), or equivalently, that \( C(A) \subset D(E) \) for all \( E \in C(A) \). Let \( E \in C(A) \). By properties (c) and (d), \( D(E) \) is a monotone
class, so it suffices to prove that \( A \subset D \). But

\[
F \in A \implies A \subset D(F) \quad \text{by the definition of } D(F), \text{ since } A \text{ is an algebra}
\]

\[
\implies C(A) \subset D(F) \quad \text{since } D(F) \text{ is a monotone class}
\]

\[
\implies E \in D(F) \quad \text{since } E \in C(A)
\]

\[
\implies F \in D(E) \quad \text{by property (b)}
\]
implies
\[
\int \nu((A \times B) \times) \, d\mu(x) = \mu(A) \nu(B) = \mu \times \nu(A \times B)
\]
\[
\int \mu((A \times B)^y) \, d\nu(x) = \mu(A) \nu(B) = \mu \times \nu(A \times B)
\]

○ \(C\) is closed under finite disjoint unions since
\[
\nu\left((\bigcup_{x} E \cup F)_{x} \right)_{\text{disjoint}} = \nu\left(E_{x} \cup F_{x}\right)_{\text{disjoint}} = \nu(E_{x}) + \nu(F_{x})
\]

Similarly for \(\mu\left((E \cup F)^y\right)\). So \(\mathcal{R} \subseteq C\).

○ \(C\) is closed under countable increasing unions:
If \(E_{1} \subseteq E_{2} \subseteq E_{3} \subseteq \cdots\) are all in \(C\) and \(E = \bigcup_{n \in \mathbb{N}} E_{n}\) then
\[
\{f_{n}(x) = \nu((E_{n})_{x})\}_{n \in \mathbb{N}} \text{ increases pointwise to } f(x) = \nu(E_{x})
\]
by continuity from below, so \(f\) is measurable and by the monotone convergence theorem
\[
\frac{f(x)}{\nu(E_{x})} \, d\mu(x) = \lim_{n \to \infty} \frac{f_{n}(x)}{\nu((E_{n})_{x})} \, d\mu(x) = \nu \mu \times \nu(E_{n}) = \mu \times \nu(E)
\]
by continuity from below. Similarly for \(\int \mu(E_{y}) \, d\nu(y)\).

○ \(C\) is closed under countable decreasing intersections:
If \(E_{1} \supset E_{2} \supset E_{3} \supset \cdots\) are all in \(C\) and \(E = \bigcap_{n \in \mathbb{N}} E_{n}\) then
\[
\{f_{n}(x) = \nu((E_{n})_{x})\}_{n \in \mathbb{N}} \text{ decreases pointwise to } f(x) = \nu(E_{x})
\]
by continuity from above. (We used \(\nu((E_{1})_{x}) \leq \nu(Y) < \infty\) here.) So \(f\) is measurable and \(0 \leq f_{n}(x) \leq f_{1}(x) \in L^{1}(X, \mathcal{M}, \mu)\) and, by the dominated convergence theorem,
\[
\int \nu(E_{x}) \, d\mu(x) = \int f(x) \, d\mu(x) \leq \lim_{n \to \infty} \int f_{n}(x) \, d\mu(x)
\]
\[
= \lim_{n \to \infty} \int \nu((E_{n})_{x}) \, d\mu(x) = \nu \mu \times \nu(E_{n})_{x} \leq \lim_{n \to \infty} \mu \times \nu(E_{n})
\]
by continuity from above. Similarly for \(\int \mu(E_{y}) \, d\nu(y)\).

So \(C\) is a monotone class that contains the algebra \(\mathcal{R}\) and hence contains
\(C(\mathcal{R}) = \mathcal{M}(\mathcal{R}) = \mathcal{M} \otimes \mathcal{N}\).
Case 2: $\mu, \nu$ $\sigma$-finite:
We can write $X \times Y$ as a countable increasing union of rectangles $\{X_n \times Y_n\}_{n \in \mathbb{N}}$ of finite measure. Then, if $E \in \mathcal{M} \otimes \mathcal{N}$, we can apply the previous case to $E \cap (X_n \times Y_n)$.

$$
\mu \times \nu (E \cap (X_n \times Y_n)) = \int \mu \left( (E \cap (X_n \times Y_n))^y \right) d\nu (y) \\
= \int \mu (E^y \cap X_n) \chi_{Y_n} (y) d\nu (y)
$$

In the limit as $n \to \infty$,
- the left hand side converges to $\mu \times \nu (E)$ by continuity from below and
- the right hand side converges to $\int \mu (E^y) d\nu (y)$ by the monotone convergence theorem.

\[ \square \]

**THEOREM 5.9** (Fubini\textsuperscript{3}–Tonelli\textsuperscript{4}). Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces.

(a) (Tonelli) If the function $f : X \times Y \to [0, \infty]$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then
- the function $g : X \to [0, \infty]$ defined by $g(x) = \int f (x, y) \, d\nu (y)$ is $\mathcal{M}$-measurable, and
- the function $h : Y \to [0, \infty]$ defined by $h(y) = \int f (x, y) \, d\mu (x)$ is $\mathcal{N}$-measurable

and

$$
\int f (x, y) \, d\mu \times \nu (x, y) = \int \left[ \int f (x, y) \, d\nu (y) \right] \, d\mu (x) \\
= \int \left[ \int f (x, y) \, d\mu (x) \right] \, d\nu (y)
$$

(b) (Fubini) If $f \in L^1(\mu \times \nu)$ then
- the function $f_x : Y \to \mathbb{R}$ defined by $f_x (y) = f (x, y)$ is in $L^1(\nu)$ for almost all $x \in X$,
- $g(x) = \int f (x, y) \, d\nu (y) \in L^1(\mu)$

\textsuperscript{3}Guido Fubini (1879–1943) was an Italian mathematician. By way of comparison, Henri Lebesgue (1875–1941) was a French mathematician and Bernhard Riemann (1826–1866) was a German mathematician.

\textsuperscript{4}Leonida Tonelli (1885–1946) was an Italian mathematician.
the function \( f^y : X \to \mathbb{R} \) defined by \( f^y(x) = f(x, y) \) is in \( L^1(\mu) \) for almost all \( y \in Y \),

\[ h(y) = \int f(x, y) \, d\mu(x) \in L^1(\nu) \]

and

\[
\int f(x, y) \, d\mu \times \nu(x, y) = \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(x)
\]

\[
= \int \left[ \int f(x, y) \, d\mu(x) \right] \, d\nu(y)
\]

**Proof.** (a) (Tonelli)

**Case 1:** \( f = \chi_E \) with \( E \in \mathcal{M} \otimes \mathcal{N} \):

This is Proposition 5.8.

**Case 2:** \( f \geq 0 \) simple:

This follows from Case 1 by linearity.

**Case 3:** \( f \in L^+(X \times Y, \mathcal{M} \otimes \mathcal{N}) \):

Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of nonnegative simple functions that increase pointwise to \( f \). For example, we could take

\[
f_n = \sum_{m=0}^{2^n-1} \frac{m}{2^n} \chi_{f^{-1}(I_{m,n})} + 2^n \chi_{f^{-1}([2^n, \infty))}(x) \quad \text{where} \quad I_{m,n} = f^{-1}(\left[ \frac{m}{2^n}, \frac{m+1}{2^n} \right])
\]

Then, by the monotone convergence theorem, the limit as \( n \to \infty \) of

\[
\int f_n(x, y) \, d\mu \times \nu(x, y) \quad \text{increases to} \quad \int f(x, y) \, d\mu \times \nu(x, y)
\]

\[
\int f(x, y) \, d\mu \times \nu(x, y) \quad \text{increases to} \quad \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(x)
\]

gives

\[
\int f(x, y) \, d\mu \times \nu(x, y) \quad \text{increases to} \quad \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(x)
\]

Similarly for the other order.
(b) (Fubini)

Write

\[ h(y) = \int f^y(x) \, d\mu(x) = \int f(x, y) \, d\mu(x) \]
\[ g(x) = \int f_x(y) \, d\nu(y) = \int f(x, y) \, d\nu(y) \]

By Tonelli,

\[ f(x, y) \in L^1 \implies \int |f(x, y)| \, d\mu \times \nu < \infty \]
\[ \implies \int \left[ \int |f(x, y)| \, d\mu(x) \right] \, d\nu(y) < \infty \]
\[ \implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } |h(y)| \leq \int |f(x, y)| \, d\mu(x) \in L^1 \]
\[ \implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } h(y) \in L^1(\nu) \]

Similarly

\[ f_x \in L^1(\nu) \text{ a.e. } x \text{ and } g(x) \in L^1(\mu) \]

Now just apply the Tonelli theorem to the positive and negative parts of \( f \), that is, to \( \max\{f(x, y), 0\} \) and \( \max\{-f(x, y), 0\} \).