5 Product Measures and Fubini-Tonelli

Our goal in this section is to prove the Fubini-Tonelli theorem, which says that, under appropriate hypotheses,

\[
\int_{X \times Y} f(x, y) \, d\mu \times \nu \, (x, y) = \int_X \left[ \int_Y f(x, y) \, d\nu(y) \right] \, d\mu(x) = \int_Y \left[ \int_X f(x, y) \, d\mu(x) \right] \, d\nu(y)
\]

5.1 Product Measures

Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces. Our first task is to define the product.

**Definition 5.1 (Product Measure).** Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be measure spaces.

(a) Define the set of finite disjoint unions of measurable rectangles in \(X \times Y\) to be

\[
\mathcal{R} = \left\{ \bigcup_{j=1}^n A_j \times B_j \mid n \in \mathbb{N}, \ A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ (A_j \times B_j) \cap (A_k \times B_k) = \emptyset, \ \text{for all} \ 1 \leq j, k \leq n \ \text{with} \ j \neq k \right\}
\]

\(\mathcal{R}\) is nonempty. We will shortly show that it is closed under complements and finite unions and so is an algebra.

(b) Define \(\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R})\) to be the \(\sigma\)-algebra generated by \(\mathcal{R}\).

(c) Define \(\pi : \mathcal{R} \rightarrow [0, \infty]\) by

\[
\pi \left( \bigcup_{j=1}^n A_j \times B_j \right) = \sum_{j=1}^n \mu(A_j) \nu(B_j)
\]

for all \(n \in \mathbb{N}\) and all \(A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ 1 \leq j \leq n\) with \((A_j \times B_j) \cap (A_k \times B_k) = \emptyset\) for all \(j \neq k\). In this definition, we use the convention that \(0 \times \infty = 0\). We will shortly show that \(\pi\) is a well-defined premeasure.

(d) Let \(\pi^*\) be the outer measure generated by \(\pi\). By Theorem 2.36,

\[
\mu \times \nu = \pi^* \upharpoonright \mathcal{M} \otimes \mathcal{N}
\]
is a measure which extends \( \pi \). We will shortly show that if \( \mu \) and \( \nu \) are \( \sigma \)-finite, then \( \mu \times \nu \) is \( \sigma \)-finite. Then it is the unique measure on \( \mathcal{M} \otimes \mathcal{N} \) such that
\[
\mu \times \nu(A \times B) = \mu(A) \nu(B) \forall A \in \mathcal{M}, \ B \in \mathcal{N}
\]

**Remark 5.2.** (a) Any finite union of measurable rectangles can also be expressed as a finite disjoint union of measurable rectangles.

So
\[
\mathcal{R} = \left\{ \bigcup_{j=1}^{n} A_j \times B_j \ \big| \ n \in \mathbb{N}, \ A_j \in \mathcal{M}, \ B_j \in \mathcal{N}, \ \text{for all} \ 1 \leq j, k \leq n \right\}
\]
too. As \((A \times B)^c = (X \times B^c) \cup (A^c \times B)\), \( \mathcal{R} \) written in this form is obviously an algebra.

(b) That \( \pi \) is a well–defined premeasure (in part (c) of the definition) is a consequence of the observation that, if
\[
\bigcup_{j=1}^{\infty} A_j \times B_j = \bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k
\]
are disjoint unions of measurable rectangles, then
\[
\sum_{k=1}^{n} \chi_{\tilde{A}_k}(x) \chi_{\tilde{B}_k}(y) = \sum_{k=1}^{n} \chi_{A_k \times B_k}(x, y) = \chi_{\bigcup_k \tilde{A}_k \times \tilde{B}_k}(x, y) = \chi_{\bigcup_j A_j \times B_j}(x, y)
\]
\[
= \sum_{j=1}^{\infty} \chi_{A_j \times B_j}(x, y) = \sum_{j=1}^{\infty} \chi_{A_j}(x) \chi_{B_j}(y)
\]

So integrating \( d\mu(x) \) gives
\[
\sum_{k=1}^{n} \mu(\tilde{A}_k) \chi_{\tilde{B}_k}(y) = \sum_{j=1}^{\infty} \mu(A_j) \chi_{B_j}(y)
\]
by the monotone convergence theorem. Then integrating $d\nu(y)$ gives
\[ \sum_{k=1}^{n} \mu(\tilde{A}_k) \nu(\tilde{B}_k) = \sum_{j=1}^{\infty} \mu(A_j) \nu(B_j) \]
\[ \pi\left( \bigcup_{k=1}^{n} \tilde{A}_k \times \tilde{B}_k \right) \]
again by the monotone convergence theorem.
(c) For the $\sigma$-finite statement in part (d) of the definition, observe that if
\[ \{A_j\}_{j \in \mathbb{N}} \subset \mathcal{M}, \{B_k\}_{k \in \mathbb{N}} \subset \mathcal{N}, X = \bigcup_{j \in \mathbb{N}} A_j, Y = \bigcup_{k \in \mathbb{N}} B_k, \mu(A_j) < \infty, \nu(B_k) < \infty \]
then
\[ \{A_j \times B_k\}_{j,k \in \mathbb{N}} \subset \mathcal{M} \otimes \mathcal{N}, X \times Y = \bigcup_{j,k \in \mathbb{N}} A_j \times B_k \]
and
\[ \mu \times \nu(A_j \times B_k) = \mu(A_j) \nu(B_k) < \infty \]

**Proposition 5.3** (Tensor product of Borel $\sigma$-algebras). Let $X$ and $Y$ be separable\(^1\) metric spaces with metrics $d_X$ and $d_Y$ respectively. Then $X \times Y$ is a metric space with metric $D((x,y),(x',y')) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}$ and $\mathcal{B}_{X \times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$. (By definition, a metric space is separable if and only if it contains a countable dense subset. For example, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, so that $\mathbb{R}$ is separable. Applying this proposition to $X = Y = \mathbb{R}$ gives that $\mathcal{B}_\mathbb{R} \otimes \mathcal{B}_\mathbb{R} = \mathcal{B}_{\mathbb{R}^2}$.)

**Proof.** Problem Set 9, #5. \(\square\)

**Proposition 5.4** (Slices — sets). Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, \(x \in X\) and \(y \in Y\).
(a) If \(E \in \mathcal{M} \otimes \mathcal{N}\), then
\[ E_x = \{ y' \in Y \mid (x, y') \in E \} \in \mathcal{N} \]
\[ E^y = \{ x' \in X \mid (x', y) \in E \} \in \mathcal{M} \]

\(^1\)For a counterexample in the nonseparable case, see Exercise 29 on page 231 of Folland.
(b) If \( f : X \times Y \to \mathbb{R} \) is \( \mathcal{M} \otimes \mathcal{N} \)-measurable, then the function \( f_x : Y \to \mathbb{R} \) defined by \( f_x(y) = f(x, y) \) is \( \mathcal{N} \)-measurable and the function \( f^y : X \to \mathbb{R} \) defined by \( f^y(x) = f(x, y) \) is \( \mathcal{M} \)-measurable.

Proof. (a) Let

\[
\mathcal{P} = \{ E \subset X \times Y \mid E_x \in \mathcal{N} \text{ for all } x \in X, \ E^y \in \mathcal{M} \text{ for all } y \in Y \}
\]

Then

- \( A \times B \in \mathcal{P} \) for all \( A \in \mathcal{M} \) and \( B \in \mathcal{N} \), since

\[
(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \quad (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases}
\]

- \( \mathcal{P} \) is closed under complements since, if \( E \in \mathcal{P} \), then

\[
(E^c)_x = \left( \bigcup_{E_x \in \mathcal{N}} E_x \right)^c \in \mathcal{N} \\
(E^c)^y = \left( \bigcup_{E^y \in \mathcal{M}} E^y \right)^c \in \mathcal{M}
\]
• $\mathcal{P}$ is closed under countable unions since, if $E_n \in \mathcal{P}$ for all $n \in \mathbb{N}$, then

$$
\left( \bigcup_n E_n \right)_x = \bigcup_{E_n \in \mathcal{N}} (E_n)_x \\
\left( \bigcup_n E_n \right)^y = \bigcup_{E_n \in \mathcal{M}} (E_n)^y
$$

So $\mathcal{P}$ is a $\sigma$-algebra which contains $\mathcal{R}$, and hence contains $\mathcal{M}(\mathcal{R}) = \mathcal{M} \otimes \mathcal{N}$.

(b) Let $B \in \mathcal{B}_\mathbb{R}$. As $f$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, $f^{-1}(B) \in \mathcal{M} \otimes \mathcal{N}$. So

$$
f_x^{-1}(B) = \{ y \in Y \mid f(x,y) = f(x,y) \in B \} = \bigcup_{E \in \mathcal{N}} f_x^{-1}(B)_x \\
(f^y)^{-1}(B) = \{ x \in X \mid f^y(x) = f(x,y) \in B \} = \bigcup_{E \in \mathcal{M}} f^y_x^{-1}(B)^y
$$

$$
\square
$$

5.2 Technical Aside — Monotone Classes

**Definition 5.5.** Let $X$ be a nonempty set. A collection $\mathcal{C} \subset \mathcal{P}(X)$ of subsets of $X$ is called a **monotone class** if

• it is closed under countable increasing unions (that is, if $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ and $E_1 \subset E_2 \subset E_3 \subset \cdots$, then $\bigcup_{n=1}^\infty E_n \in \mathcal{C}$) and

• it is closed under countable decreasing intersections (that is, if $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{C}$ and $E_1 \supset E_2 \supset E_3 \supset \cdots$, then $\bigcap_{n=1}^\infty E_n \in \mathcal{C}$).

**Remark 5.6.**

(a) Monotone classes are closely related to $\sigma$–algebras. In fact, for us, their only use will be to help verify that a certain collection of subsets is a $\sigma$–algebra.

(b) Every $\sigma$–algebra is a monotone class, because $\sigma$–algebras are closed under arbitrary countable unions and intersections.

(c) If, for every index $i$ in some index set $\mathcal{I}$, $\mathcal{C}_i$ is a monotone class, then $\bigcap_{i \in \mathcal{I}} \mathcal{C}_i$ is also a monotone class. In particular, for any $\mathcal{E} \subset \mathcal{P}(X)$, the collection

$$
\mathcal{C}(\mathcal{E}) = \bigcap_{\mathcal{C} \text{ monotone class} \subset \mathcal{E}} \mathcal{C}
$$

Product Measures and Fubini-Tonelli

November 12, 2019
is a monotone class, called the monotone class generated by $E$. It is the smallest
monotone class that contains $E$. So if $C$ is any monotone class that contains $E,$
then $C(E) \subseteq C$.

**Lemma 5.7.** Let $X$ be a nonempty set. If $A \subseteq \mathcal{P}(X)$ is an algebra, then
\[
\mathcal{C}(A) = \mathcal{M}(A)
\]
That is, the monotone class generated by $A$ is the same as the $\sigma$–algebra generated
by $A$.

**Proof.**
- $\mathcal{C}(A) \subseteq \mathcal{M}(A)$:
  \[\mathcal{M}(A)	ext{ is a }\sigma\text{–algebra, and hence a monotone class, that contains }A.\text{ So, this}
  \text{follows by part (c) of Remark 5.6.}\]
- $\mathcal{M}(A) \subseteq \mathcal{C}(A)$:
  It suffices to prove that $\mathcal{C}(A)$ is a $\sigma$–algebra, because then we will know that
  $\mathcal{C}(A)$ is a $\sigma$–algebra containing $A$ and hence $\mathcal{M}(A)$, which is the smallest
  $\sigma$–algebra containing $A$.
  By Problem Set 1, # 6, any algebra that is closed under countable increasing
  unions is a $\sigma$–algebra. So it suffices to prove that $\mathcal{C}(A)$ is an algebra (i.e. that
  $\mathcal{C}(A)$ is nonempty and closed under complements and finite intersections). So
  it suffices to prove
  \[E, F \in \mathcal{C}(A) \implies E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(A) \quad \text{(*)}\]
  (since $X$ is automatically in $A$, which is an algebra, and hence is automatically
  in $\mathcal{C}(A)$ and is an allowed choice for $E$). Define, for each $E \in \mathcal{C}(A),$
  \[\mathcal{D}(E) = \{ F \in \mathcal{C}(A) \mid E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(A) \} \]
  We wish to show that
  \[E \in \mathcal{C}(A) \implies \mathcal{C}(A) \subseteq \mathcal{D}(E)\]
  To do so, it suffices to show that $\mathcal{D}(E)$ is a monotone class that contains $A$.
  We first prove some properties of $\mathcal{D}(E)$.
  (a) $E \in \mathcal{C}(A) \implies \emptyset, E \in \mathcal{D}(E)$.
  (b) For $E, F \in \mathcal{C}(A),$
  \[F \in \mathcal{D}(E) \iff E \setminus F, F \setminus E, E \cap F \in \mathcal{C}(A)\]
  \[\iff E \in \mathcal{D}(F) = \{ F' \in \mathcal{C}(A) \mid F \setminus F', F' \setminus F, F \cap F' \in \mathcal{C}(A) \}\]
(c) \(D(E)\) is closed under countable increasing unions. To see this, let the sequence \(\{F_n\}_{n \in \mathbb{N}} \subset D(E)\) obey \(F_1 \subset F_2 \subset F_3 \subset \cdots\) and set \(F = \bigcup_{n=1}^{\infty} F_n\). Then
\[
\text{· } \{E \setminus F_n = E \cap F_n^c\}_{n \in \mathbb{N}} \subset \mathcal{C}(A) \text{ is decreasing,}
\]
\[
\text{· } \{F_n \setminus E = F_n \cap E^c\}_{n \in \mathbb{N}} \subset \mathcal{C}(A) \text{ is increasing and}
\]
\[
\text{· } \{E \cap F_n\}_{n \in \mathbb{N}} \subset \mathcal{C}(A) \text{ is increasing,}
\]
so that
\[
E \setminus F = E \cap \left(\bigcup_{n=1}^{\infty} F_n\right)^c = E \cap \left(\bigcap_{n=1}^{\infty} F_n^c\right) = \bigcap_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(A)
\]
\[
F \setminus E = \left(\bigcup_{n=1}^{\infty} F_n\right) \cap E^c = \bigcup_{n=1}^{\infty} (F_n \cap E^c) = \bigcup_{n=1}^{\infty} (F_n \setminus E) \in \mathcal{C}(A)
\]
\[
E \cap F = E \cap \left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(A)
\]
since \(\mathcal{C}(A)\) is closed under countable decreasing intersections and countable increasing unions. So \(F \in D(E)\).

(d) \(D(E)\) is closed under countable decreasing intersections. To see this, let \(\{F_n\}_{n \in \mathbb{N}} \subset D(E)\) obey \(F_1 \supset F_2 \supset F_3 \supset \cdots\) and set \(F = \bigcap_{n=1}^{\infty} F_n\). As in part (c)
\[
E \setminus F = E \cap \left(\bigcap_{n=1}^{\infty} F_n\right)^c = E \cap \left(\bigcup_{n=1}^{\infty} F_n^c\right) = \bigcup_{n=1}^{\infty} (E \cap F_n^c) \in \mathcal{C}(A)
\]
\[
F \setminus E = \left(\bigcap_{n=1}^{\infty} F_n\right) \cap E^c = \bigcap_{n=1}^{\infty} (F_n \cap E^c) = \bigcap_{n=1}^{\infty} (F_n \setminus E) \in \mathcal{C}(A)
\]
\[
E \cap F = E \cap \left(\bigcap_{n=1}^{\infty} F_n\right) = \bigcap_{n=1}^{\infty} (E \cap F_n) \in \mathcal{C}(A)
\]
so \(F \in D(E)\).

We are now ready to prove (*), or equivalently, that \(\mathcal{C}(A) \subset D(E)\) for all \(E \in \mathcal{C}(A)\). Let \(E \in \mathcal{C}(A)\). By properties (c) and (d), \(D(E)\) is a monotone
class, so it suffices to prove that $A \subset D$. But

$$F \in A \implies A \subset D(F) \quad \text{by the definition of } D(F), \text{ since } A \text{ is an algebra}$$

$$\implies C(A) \subset D(F) \quad \text{since } D(F) \text{ is a monotone class}$$

$$\implies E \in D(F) \quad \text{since } E \in C(A)$$

$$\implies F \in D(E) \quad \text{by property (b)}$$

\[ \square \]

5.3 Fubini-Tonelli

Proposition 5.8 (Slices — measure). Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces and let $E \in \mathcal{M} \otimes \mathcal{N}$. Then the function $f : X \to [0, \infty]$ defined by $f(x) = \nu(E_x)$ is $\mathcal{M}$-measurable and the function $g : Y \to [0, \infty]$ defined by $g(y) = \mu(E^y)$ is $\mathcal{N}$-measurable and

$$\mu \times \nu(E) = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y)$$

Proof.

Case 1: $\mu, \nu$ finite:

Set

$$C = \{ \ E \in \mathcal{M} \otimes \mathcal{N} \mid \text{conclusions of the theorem are true} \}$$

We will show that

$C$ is a monotone class that contains $\mathcal{R}$ (the set of finite unions of measurable rectangles)

which will then imply that

$$\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R}) = C(\mathcal{R}) \subset C$$

We have

- $A \in \mathcal{M}, \ B \in \mathcal{N} \implies A \times B \in C$ since

  $$\nu((A \times B)_x) = \nu\left(\begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}\right) = \chi_A(x) \nu(B)$$

  $$\mu((A \times B)^y) = \chi_B(y) \mu(A)$$
implies
\[ \int \nu((A \times B)_x) \, d\mu(x) = \mu(A) \nu(B) = \mu \times \nu(A \times B) \]
and
\[ \int \mu((A \times B)^y) \, d\nu(x) = \mu(A) \nu(B) = \mu \times \nu(A \times B) \]

- **C** is closed under finite disjoint unions since
  \[ \nu\left(\bigcup_{\text{disjoint}} (E \cup F)_x\right) = \nu\left((E_x \cup F_x)_x\right) = \nu(E_x) + \nu(F_x) \]

Similarly for \( \mu\left(\bigcup_{\text{disjoint}} (E \cup F)_x\right) \). So \( \mathcal{R} \subseteq \mathcal{C} \).

- **C** is closed under countable increasing unions:
  If \( E_1 \subset E_2 \subset E_3 \subset \cdots \) are all in \( \mathcal{C} \) and \( E = \bigcup_{n \in \mathbb{N}} E_n \) then
  \[ \{ f_n(x) = \nu\left(\bigcup_{n \in \mathbb{N}} (E_n)\right) \}_{n \in \mathbb{N}} \]
  increases pointwise to \( f(x) = \nu(E_x) \)
  by continuity from below, so \( f \) is measurable and by the monotone convergence theorem
  \[ \int \nu(E_x) \, d\mu(x) = \lim_{n \to \infty} \int \nu\left(\bigcup_{n \in \mathbb{N}} (E_n)\right) \, d\mu(x) \leq \lim_{n \to \infty} \mu \times \nu(E_n) = \mu \times \nu(E) \]
  by continuity from below. Similarly for \( \int \mu(E^y) \, d\nu(y) \).

- **C** is closed under countable decreasing intersections:
  If \( E_1 \supset E_2 \supset E_3 \supset \cdots \) are all in \( \mathcal{C} \) and \( E = \bigcap_{n \in \mathbb{N}} E_n \) then
  \[ \{ f_n(x) = \nu\left(\bigcap_{n \in \mathbb{N}} (E_n)\right) \}_{n \in \mathbb{N}} \]
  decreases pointwise to \( f(x) = \nu(E_x) \)
  by continuity from above. (We used \( \nu(E_1) < \infty \) here.) So \( f \) is measurable and \( 0 \leq f_n(x) \leq f_1(x) \in L^1(\mathcal{X}, \mathcal{M}, \mu) \) and, by the dominated convergence theorem,
  \[ \int \nu(E_x) \, d\mu(x) = \int f(x) \, d\mu(x) = \lim_{n \to \infty} \int f_n(x) \, d\mu(x) \]
  \[ = \lim_{n \to \infty} \int \nu\left(\bigcap_{n \in \mathbb{N}} (E_n)\right) \, d\mu(x) \leq \lim_{n \to \infty} \mu \times \nu(E_n) \]
  \[ = \mu \times \nu(E) \]
  by continuity from above. Similarly for \( \int \mu(E^y) \, d\nu(y) \).

So \( \mathcal{C} \) is a monotone class that contains the algebra \( \mathcal{R} \) and hence contains
\( \mathcal{C}(\mathcal{R}) = \mathcal{M}(\mathcal{R}) = \mathcal{M} \times \mathcal{N} \).
Case 2: $\mu, \nu$ $\sigma$-finite:

We can write $X \times Y$ as a countable increasing union of rectangles $\{X_n \times Y_n\}_{n \in \mathbb{N}}$ of finite measure. Then, if $E \in \mathcal{M} \otimes \mathcal{N}$, we can apply the previous case to $E \cap (X_n \times Y_n)$.

$$\mu \times \nu(E \cap (X_n \times Y_n)) = \int \mu((E \cap (X_n \times Y_n))^y) \, d\nu(y)$$

$$= \int \mu(E^y \cap X_n) \chi_{Y_n}(y) \, d\nu(y)$$

In the limit as $n \to \infty$,

- the left hand side converges to $\mu \times \nu(E)$ by continuity from below and
- the right had side converges to $\int \mu(E^y) \, d\nu(y)$ by the monotone convergence theorem.

THEOREM 5.9 (Fubini\textsuperscript{2}-Tonelli\textsuperscript{3}). Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces.

(a) (Tonelli) If the function $f : X \times Y \to [0, \infty]$ is $\mathcal{M} \otimes \mathcal{N}$-measurable, then

- the function $g : X \to [0, \infty]$ defined by $g(x) = \int f(x, y) \, d\nu(y)$ is $\mathcal{M}$-measurable, and
- the function $h : Y \to [0, \infty]$ defined by $h(y) = \int f(x, y) \, d\mu(x)$ is $\mathcal{N}$-measurable

and

$$\int f(x, y) \, d\mu \times \nu(x, y) = \int \left[ \int f(x, y) \, d\nu(y) \right] \, d\mu(x)$$

$$= \int \left[ \int f(x, y) \, d\mu(x) \right] \, d\nu(y)$$

(b) (Fubini) If $f \in L^1(\mu \times \nu)$ then

- the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ is in $L^1(\nu)$ for almost all $x \in X$,
- $g(x) = \int f(x, y) \, d\nu(y) \in L^1(\mu)$
- the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x, y)$ is in $L^1(\mu)$ for almost all $y \in Y$,
- $h(y) = \int f(x, y) \, d\mu(x) \in L^1(\nu)$

\textsuperscript{2}Guido Fubini (1879-1943) was an Italian mathematician

\textsuperscript{3}Leonida Tonelli (1885-1946) was an Italian mathematician

Product Measures and Fubini-Tonelli 11

November 12, 2019
and
\[ \int f(x, y) \, d\mu \times \nu(x, y) = \int \left[ \int f(x, y) \, d\nu(y) \right] d\mu(x) \]
\[ = \int \left[ \int f(x, y) \, d\mu(x) \right] d\nu(y) \]

\textbf{Proof.} (a) (Tonelli)

\textit{Case 1:} \( f = \chi_E \) with \( E \in \mathcal{M} \otimes \mathcal{N} \):

This is Proposition 5.8.

\textit{Case 2:} \( f \geq 0 \) simple:

This follows from Case 1 by linearity.

\textit{Case 3:} \( f \in L^+(X \times Y, \mathcal{M} \otimes \mathcal{N}) \):

Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of nonnegative simple functions that increase pointwise to \( f \). For example, we could take

\[ f_n = \sum_{m=0}^{2^n - 1} \frac{m}{2^n} \chi_f^{-1}(I_{m,n}) + 2^n \chi_f^{-1}([2^n, \infty]) \]

where \( I_{m,n} = f^{-1}(\left[ \frac{m}{2^n}, \frac{m+1}{2^n} \right]) \).

Then, by the monotone convergence theorem

\[ \int f_n(x, y) \, d\mu \times \nu(x, y) \rightarrow \int f(x, y) \, d\mu \times \nu(x, y) \]

\[ \int \int f_n(x, y) \, d\mu(x) \, d\nu(y) \rightarrow \int \int f(x, y) \, d\mu(x) \, d\nu(y) \]

and similarly for the other order.

(b) (Fubini)

Write

\[ h(y) = \int f^y(x) \, d\mu(x) = \int f(x, y) \, d\mu(x) \]
\[ g(x) = \int f_x(y) \, d\nu(y) = \int f(x, y) \, d\nu(y) \]

Product Measures and Fubini-Tonelli

November 12, 2019
By Tonelli,

\[ f(x, y) \in L^1 \implies \int |f(x, y)| \, d\mu \times \nu < \infty \]

\[ \implies \int \left[ \int |f(x, y)| \, d\mu(x) \right] \, d\nu(y) < \infty \]

\[ \implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } |h(y)| \leq \int |f(x, y)| \, d\mu(x) \in L^1 \]

\[ \implies f^y \in L^1(\mu) \text{ a.e. } y \text{ and } h(y) \in L^1(\nu) \]

Similarly

\[ f_x \in L^1(\nu) \text{ a.e. } x \text{ and } g(x) \in L^1(\mu) \]

Now just apply Tonelli to the positive and negative parts of \( f \).