Math 420 Problem Set 8
Due Wednesday, November 6

1. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f : X \to [0, \infty)\) a measurable function. Define, for each \(E \in \mathcal{M}\),
\[
\lambda(E) = \int_E f \, d\mu.
\]
Prove that \(\lambda\) is a measure on \(\mathcal{M}\) and that \(\int g \, d\mu = \int g \, d\lambda\) for all measurable functions \(g : X \to [0, \infty)\).

2. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f_n : X \to [0, \infty)\) be a sequence of measurable functions that decreases to the function \(f\). Prove that if \(\int f_1 \, d\mu < \infty\), then \(\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu\). Is the implication still valid when \(\int f_1 \, d\mu = \infty\)?

3. Prove the following variant of Fatou’s lemma:
Let \((X, \mathcal{M}, \mu)\) be a measure space and \(E \in \mathcal{M}\). Let \(g \in L^1(X)\) be nonnegative. Assume that, for each \(n \in \mathbb{N}\), \(f_n : X \to \mathbb{R}\) is measurable and \(f_n(x) \geq -g(x)\) for all \(x \in E\). Then
\[
\int_E \liminf_{n \to \infty} f_n(x) \, d\mu(x) \leq \liminf_{n \to \infty} \int_E f_n(x) \, d\mu(x).
\]
What is the analog of Fatou’s lemma for nonpositive functions?

Note: In class, we defined \(\int f(x) \, d\mu(x)\) only for \(f \in L^1(X, \mu)\) and for \(f \geq 0\) measurable. The \(f_n\)’s in this problem need not be \(L^1\) and need not nonnegative. Part of this problem is to extend the definition of \(\int f(x) \, d\mu(x)\) to a class of functions that include the \(f_n\)’s.

4. Compute the following limits and justify the calculations.
   (a) \(\lim_{n \to \infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin \frac{x}{n} \, dx\)
   (b) \(\lim_{n \to \infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx\)
   (c) \(\lim_{n \to \infty} \int_0^\infty \frac{1}{x(1+x^n)} \sin \frac{x}{n} \, dx\)
   (d) \(\lim_{n \to \infty} \int_0^\infty \frac{n}{1+n^2x^2} \, dx\) (The answer depends on whether \(a > 0\), \(a = 0\) or \(a < 0\). How does this accord with the various convergence theorems?)

5. Let \(f \in L^1(\mathbb{R}, m)\), with \(m\) being Lebesgue measure, and set \(F(x) = \int_{-\infty}^x f(t) \, dm(t)\). Prove that \(F\) is continuous on \(\mathbb{R}\).