Math 420 Problem Set 7
Due Wednesday, October 30

1. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f, g : X \to \mathbb{R}\) or \(f, g : X \to [0, \infty]\) be measurable functions with \(f = g\) almost everywhere. Let \(E \in \mathcal{M}\).
   (a) Prove that if \(f, g \geq 0\), then \(\int_E f \, d\mu = \int_E g \, d\mu\).
   (b) Prove that if \(f \in L^1\), then \(g \in L^1\) and \(\int_E f \, d\mu = \int_E g \, d\mu\).

2. Let \(\mu\) be a finite measure. Prove the inclusion-exclusion formula:

\[
\mu\left(\bigcup_{i=1}^n A_i\right) = \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|S|+1} \mu\left(\bigcap_{i \in S} A_i\right)
\]

Here \(|S|\) is the number of elements of the set \(S\). (Hint: Integrate the function \(1 - \prod_{i=1}^n (1 - \chi_{A_i})\).)

3. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f : X \to [0, \infty)\) be a measurable function with \(\int f \, d\mu < \infty\).

   Prove that, for every \(\varepsilon > 0\), there exists an \(E \in \mathcal{M}\) with \(\mu(E) < \varepsilon\) and \(\int f \, d\mu \leq \int_E f \, d\mu + \varepsilon\).

4. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f : X \to [0, \infty)\) be a nonnegative measurable function with \(\int f \, d\mu < \infty\).

   Prove that \(\{ x \in X \mid f(x) > 0 \}\) is \(\sigma\)-finite.

5. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let, for each \(n \in \mathbb{N}\), \(f_n : X \to [0, \infty)\) be measurable. Suppose that \(f_n \to f\) pointwise and that \(\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu\).

   (a) Prove that if \(\int f \, d\mu < \infty\), then \(\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu\) for all \(E \in \mathcal{M}\).

   (b) Find an example for which \(\int f \, d\mu = \infty\) and \(\int_E f \, d\mu \neq \lim_{n \to \infty} \int f_n \, d\mu\) for some \(E \in \mathcal{M}\).

6. Let \(\varepsilon > 0\) and \(-\infty < a < b < \infty\). Let \(m\) be Lebesgue measure and \(f : [a, b] \to \mathbb{R}\) be a Lebesgue-measurable function.

   (a) Prove that there exists an \(M \in [0, \infty)\) such that

   \[
m\{ x \in [a, b] \mid |f(x)| \geq M \} \leq \varepsilon
   \]

   (b) Assume that \(f : [a, b] \to [c, C]\). Prove that there exists a simple function \(\sigma\) such that \(c \leq \sigma(x) \leq C\) and \(|f(x) - \sigma(x)| \leq \varepsilon\) for all \(x \in [a, b]\). A simple function is, by definition, of the form \(\sum_{j=1}^n a_j \chi_{E_j}(x)\) with \(n \in \mathbb{N}\) and the sets \(E_j\) measurable.

   (c) Let \(\sigma : [a, b] \to [c, C]\) be a simple function. Prove that there exists a step function \(s : [a, b] \to [c, C]\) such that the measure

   \[
m\{ x \in [a, b] \mid \sigma(x) \neq s(x) \} < \varepsilon
   \]

   A step function is, by definition, of the form \(\sum_{i=1}^n a_i \chi_{E_i}(x)\) with \(n \in \mathbb{N}\) and the sets \(E_i\) intervals.

   (d) Let \(s : [a, b] \to [c, C]\) be a step function. Prove that there exists a continuous function \(g : [a, b] \to [c, C]\) such that

   \[
m\{ x \in [a, b] \mid s(x) \neq g(x) \} < \varepsilon
   \]

Remark: Here is the point of problem 6. If we think of functions that differ only on sets of measure zero as being the same function (this idea can easily be implemented rigorously by using equivalence classes), then \(L^1([a, b])\) is naturally a metric space with metric \(d(f, g) = \int_a^b |f(x) - g(x)| \, dx\). Using Problem 6, it is easy to show that the set of continuous functions is dense in \(L^1([a, b])\). (Ask me if you don’t see this.) With one more simple trick (ask me if you want to see it) one can also show that \(C^\infty\) functions are dense in \(L^1\).