Math 420 Problem Set 6
Due Wednesday, October 23

1. Let $m$ be the Lebesgue measure on $\mathbb{R}$, $m^*$ be the corresponding outer measure and $\mathcal{L}$ be the collection of Lebesgue measurable sets. Define, for any $E \subset \mathbb{R}$ and any $t \in \mathbb{R}$, the sets $E + t = \{ x + t \mid x \in E \}$ and $tE = \{ tx \mid x \in E \}$.
   (a) Prove that if $E \subset \mathbb{R}$ and $t \in \mathbb{R}$, then $m^*(E + t) = m^*(E)$ and $m^*(tE) = |t| m^*(E)$.
   (b) Prove that if $E \in \mathcal{L}$ and $t \in \mathbb{R}$, then $E + t \in \mathcal{L}$, $tE \in \mathcal{L}$, $m(E + t) = m(E)$ and $m(tE) = |t| m(E)$.

2. Let $X$ and $Y$ be nonempty sets and let $\mathcal{N}$ be a $\sigma$-algebra on $Y$ and let $f : X \to Y$. Prove that $\mathcal{M} = \{ f^{-1}(N) \mid N \in \mathcal{N} \}$ is a $\sigma$-algebra and is the smallest $\sigma$-algebra on $X$ with respect to which $f$ is measurable. It is sometimes called the $\sigma$-algebra generated by $f$.

3. Let $\{X, \mathcal{M}\}$ be a measure space and $f : X \to \mathbb{R}$. Prove that $f$ is measurable if and only if $f^{-1}\left((q, \infty)\right) \in \mathcal{M}$ for all $q \in \mathbb{Q}$.

4. Let $\{X, \mathcal{M}\}$ be a measure space. Find sufficient conditions on $\{X, \mathcal{M}\}$ such that there exists a bounded family of measurable functions $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in \mathcal{I}}$ whose supremum is not measurable. ($\mathcal{I}$ will be uncountable.)

5. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then it is Borel measurable.

6. Let $\{X, \mathcal{M}\}$ be a measure space and $\{f_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ a sequence of measurable functions. Prove that $\{ x \in X \mid \lim_{n \to \infty} f_n(x) \text{ exists} \} \in \mathcal{M}$.

7. Let $X$ be a nonempty set and $\mathcal{M}$ a $\sigma$-algebra of subsets of $X$. A function $f : X \to \mathbb{R}$ is said to be measurable on $A \in \mathcal{M}$ if $f^{-1}(B) \cap A \in \mathcal{M}$ for all Borel sets $B$. Equivalently, $f$ is measurable on $A$ if the restriction of $f$ to $A$ is $\mathcal{M}_A$-measurable, where $\mathcal{M}_A = \{ E \cap A \mid E \in \mathcal{M} \}$. Let $A, B \in \mathcal{M}$ with $X = A \cup B$. Prove that $f : X \to \mathbb{R}$ is measurable if and only if $f$ is measurable on $A$ and on $B$.

8. Let $X$ be a nonempty set and $\mathcal{M}$ a $\sigma$-algebra of subsets of $X$. Define $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $\mathcal{B}_{\overline{\mathbb{R}}} = \{ E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_\mathbb{R} \}$.
   (a) Prove that $\mathcal{B}_{\overline{\mathbb{R}}}$ is a $\sigma$-algebra.
   (b) Prove that $f : X \to \overline{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable if and only if $f^{-1}(\{\infty\}) \in \mathcal{M}$, $f^{-1}(\{-\infty\}) \in \mathcal{M}$ and $f$ is measurable on $f^{-1}(\mathbb{R})$.

9. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that the following implications are true if and only if $\mu$ is complete.
   (a) Let $f, g : X \to \mathbb{R}$. If $f$ is measurable and $f = g$ $\mu$-a.e., then $g$ is measurable.
   (b) Let $f : X \to \mathbb{R}$ and $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$. If $f_n$ is measurable for all $n \in \mathbb{N}$ and $\{f_n\}$ converges $\mu$-a.e. to $f$, then $f$ is measurable.