Math 420 Problem Set 4  
Due Wednesday, October 9

1. Let $A \subset \mathcal{P}(X)$ be an algebra, $\mu_0$ be a premeasure on $A$ and $\mu^*$ the induced outer measure.
   
   (a) Denote by $A_\sigma$ the collection of countable unions of sets in $A$. Prove that, for any $E \subset X$ and $\varepsilon > 0$, there exists an $A \in A_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.
   
   (b) Denote by $A_{\sigma \delta}$ the collection of countable intersections of sets in $A_\sigma$. Prove that if $E$ is any subset of $X$ with $\mu^*(E) < \infty$, then $E$ is $\mu^*$-measurable if and only if there exists a $B \in A_{\sigma \delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
   
   (c) Prove that if $\mu_0$ is $\sigma$-finite, then the restriction $\mu^*(E) < \infty$ of part (b) is superfluous.

2. Let $\mu^*$ be an outer measure induced on $X$ from a finite premeasure $\mu_0$. Define, for each $E \subset X$, the inner measure of $E$ to be $\mu^*(E) = \mu_0(X) - \mu^*(E^c)$. Prove that $E$ is $\mu^*$-measurable if and only if $\mu^*(E) = \mu^*(E)$.

3. Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, $\mu^*$ the outer measure induced from $\mu$ and $\bar{\mu}$ the restriction of $\mu^*$ to $\mathcal{M}^*$, the collection of $\mu^*$-measurable sets. Prove that $\bar{\mu}$ is the completion of $\mu$.

4. Let $A$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ with $-\infty \leq a < b \leq \infty$. (When $b = \infty$, interpret $(a, b]$ as $(a, \infty]$.)
   
   (a) Prove that $A$ is an algebra on $\mathbb{Q}$.
   
   (b) Prove that the $\sigma$-algebra generated by $A$ is $\mathcal{P}(\mathbb{Q})$. (Recall that $\mathcal{P}(\mathbb{Q})$ is the set of all subsets of $\mathbb{Q}$.)
   
   (c) Define $\mu_0$ on $A$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Prove that $\mu_0$ is a premeasure on $A$ and that there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $A$ is $\mu_0$.

5. Let $F$ be a nondecreasing and right continuous function on $\mathbb{R}$ and let $\mu_F$ be a measure that obeys $\mu_F((a, b]) = F(b) - F(a)$ for all $-\infty < a < b < \infty$. Define $F(c^-)$ to be the limit of $F(x)$ as $x$ approaches $c$ from the left. That is,
   
   $$F(c^-) = \lim_{x \downarrow c} F(x)$$
   
   Prove that, for all $-\infty < a < b < \infty$,
   
   $$\mu_F((a]) = F(a) - F(a^-) \quad \mu_F((a, b]) = F(b) - F(a^-)$$
   
   $$\mu_F([a, b)) = F(b^-) - F(a^-) \quad \mu_F([a, b]) = F(b) - F(a^-)$$